Linear Interest Rate Derivatives

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Summary (1/2)

- The discount factor
- Interest-rates: definitions
- Compounding conventions: simple, compounded, continuous
- Money market and the discount factor
- Yield curves described by a single rate
- Interbank deposits and deposit rates

Summary (2/2)

- LIBOR rates
- Forward-rate agreements (FRAs)
- Interest-rate futures
- Definition of schedule and IMM dates
- Interest-rates swaps
- IRS fixed leg, IBOR leg, and fair swap rate

Interest-rate compounding conventions

Specify how to compute the coupon cash value, $C = N\,c$, given

- A notional value N
- An interest rate r
- An initial coupon date d_1 and a final coupon date d_2
- The standard conventions (day-count, end-of-month, adjustment)

Simple compounding

The basic, natural, compounding: no compounding. Used for small time periods, when interests on interests are not considered.

$$c = r\,T(d_1, d_2) \tag{1}$$

Example:
Interbank deposit rates are quoted this way

Annual compounding

Used for multi-year interest rates. Interests are compounded every year for n years:

$$c = (1+r)^n - 1 = (1+r)^{T(d_1,d_2)} - 1 \tag{2}$$

E.g., simple compounding at r for two years in a row (assume T=1)

$$\begin{aligned} R_{1Y} &= N + N\,r = N(1+r) \\ R_{2Y} &= R_{1Y} + R_{1Y}\,r = R_{1Y} * (1+r) \\ &= N(1+r) * (1+r) = N(1+r)^2 \end{aligned} \tag{3}$$

Period compounding

Compounding over periods shorter than one year

$$c = \left(1 + \frac{r}{k}\right)^{k\,T(d_1,d_2)} - 1 \qquad (4)$$

with k the number of periods in a year

E.g., for k=2 compounding is twice a year,

$$c = \left(1 + \frac{r}{2}\right)^{2\,T(d_1,d_2)} - 1 \qquad (5)$$

Continuously compounding

This is the *quant* compounding, misunderstood by some in the industry. It can be achieved by compounding over very small time periods. In the limit for $k \to \infty$, we have

$$c = e^{rT(d_1,d_2)} - 1 \tag{6}$$

Usually used in sophisticated short-term interest-rate models. Convenient at all time scales

Exercise: Show that

$$\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{k\,T(d_1,d_2)} = e^{rT(d_1,d_2)} \tag{7}$$

Standard compounding

Used to quotes bond yield curves: simple compounding for periods smaller than one year, annual for longer periods

Note: for periods longer than one year

$$\text{Simple} \leq \text{Annual} \leq \text{Period} \leq \text{Continuous}$$

The money-market account (1/3)

We invest at d_0 an amount of N units of currency in the *money market* at an over-night interest rate r_1. The day after, at d_1, we have

$$M(d_1) = N(1 + r_1\,\tau_1) \qquad \text{with} \quad \tau_1 = T(d_0, d_1) \tag{8}$$

On the same day, we invest all the money at the over-night rate r_1, so that on the third day we have

$$M(d_2) = N(1 + r_1\,\tau_1)(1 + r_2\,\tau_2) \qquad \text{with} \quad \tau_2 = T(d_1, d_2) \tag{9}$$

The money-market account (2/3)

Each day invest all the money in the money market until d_n,

$$M(d_n) = N(1 + r_1\,\tau_1)(1 + r_2\,\tau_2)\ldots(1 + r_n\,\tau_n) \tag{10}$$

Since the numbers τ_i's are small we have

$$(1 + r_i\,\tau_i) \simeq e^{r_i\,\tau_i} \qquad \text{for } i = 1, \ldots, n \tag{11}$$

So that

$$M(d_n) \simeq N\,e^{r_1\,\tau_1} e^{r_2\,\tau_2} \ldots e^{r_n\,\tau_n} \tag{12}$$

The money-market account (3/3)

Using the properties of the exponential function we have,

$$M(d_n) \simeq N\, e^{r_1\,\tau_1 + r_2\,\tau_2 + \ldots + r_n\,\tau_n} \tag{13}$$

In the limit of smaller and smaller compounding, for a generic date d, we can define the money-market account to grow as

$$M(d) = N\, e^{\int_0^{T(d_0,d)} r(t)\mathrm{d}t} \tag{14}$$

The money-market account is stochastic!

Questions?

Discount factor (1/3)

The value of a unit of currency is not constant in time

- It is possible to obtain future interests by lending an amount of cash
- Holding cash gives us a liquidity convenience
- The total amount of a currency that is available changes with its money supply

The discount factor is the most important concept in quantitative finance

Discount factor (2/3)

The discount factor is the present value of a future unit of currency

$$D(d) = \text{PV}(1 \text{ paid at } d) \tag{15}$$

with PV denoting the present-value operator

Note that, when interest-rates are positive, $d_1 < d_2$ implies

$$D(d_1) > D(d_2) \tag{16}$$

Discount factor (3/3)

Theorem:
The expected value of the inverse money-market account, in the risk-neutral measure (i.e. when there is no arbitrage), is given by

$$E\left[\frac{1}{M(d)}\right] = D(d) \tag{17}$$

Recall that while $M(d)$ is a stochastic quantity, $D(d)$ is well defined and deterministic.

Using equation (14) we have,

$$D(d) = E\left[e^{-r_1\,\tau_1 - \ldots - r_n\,\tau_n}\right] = E\left[e^{-\int_0^{T(d)} r(t)\mathrm{d}t}\right] \tag{18}$$

Flat term structure (1/3)

Given an interest rate z (+ conventions) and a reference date d_0

$$\text{Future value:} \qquad 1 = N(1 + c) \tag{19}$$

where c is computed from d_0 to a future date d

The present value of $N(1 + c)$ is N

$$\text{Present value:} \qquad N = \frac{1}{1 + c} = D(d) \tag{20}$$

by the definition of discount factor

Flat term structure (2/3)

Simple compounding

$$D(d) = \frac{1}{1 + z\,T(d_0, d)} \tag{21}$$

Annual compounding

$$D(d) = \frac{1}{(1 + z)^{T(d_0,d)}} \tag{22}$$

Continuous compounding

$$D(d) = e^{-z\,T(d_0,d)} \tag{23}$$

The interest rate z is sometimes called the *zero rate*

Flat term structure (3/3)

Discount factor with continuous compounding

$$D(d) = e^{-z\,T(d_0,d)} \tag{24}$$

Equation (18) becomes

$$e^{-z\,T(d_0,d)} = E\left[e^{-\int_{d_0}^{T(d)} r(t)\mathrm{d}t}\right] \tag{25}$$

so that

$$z = \frac{-1}{T(d_0,d)} \log\left(E\left[e^{-\int_{d_0}^{T(d)} r(t)\mathrm{d}t}\right]\right) \tag{26}$$

Present value of an instrument

Consider a financial instrument with cash flows c_1, c_2, ..., c_n at future dates d_1, d_2, ..., d_n

$$\text{PresentValue} = D(d_1)\, c_1 + D(d_2)\, c_2 + \ldots + D(d_n)\, c_n \tag{27}$$

- no credit risk
- dates are deterministic
- cash flows are known today

Arbitrage and risk-free discount factor

Theorem:
There is only one risk-free discount curve

Proof: Otherwise two different risk-free investments over the same time period would yield two different interest payments and somebody could generate unlimited cash by buying one and selling the other

Note: The discount curve of tomorrow, or at any future date, is not unique (or known)

Questions?

Interbank loans and deposit rates

- Banks sometimes require more liquidity than they have access to, while some other days they have a surplus of cash
- A “lively” inter-bank lending market has therefore developed
- At any given time the deposit rate is the interest rate at which big banks lend to each other (i.e. intra-day quotes are available)
- Deposit maturities range from one day to one year

Settlement days

Typically an inter-bank deposit, i.e. a loan from one bank to the other, is agreed today, however money are exchanged sd days later

When	Today	Settlement date	Maturity Date
Date	T = 0	S = T + 2 days	M = S + 3 months
Cash-flow	0	N	$N(1 + r\,\tau)$

Deposit in EUR/USD longer than 1 day have usually two days of settlement

Deposit in GBP have usually zero days of settlement

LIBOR fixings

The average deposit rate at which big banks lend money

British Bankers' Association publishes the LIBOR rate: the *London Interbank Bank Offer Rate*

For Euro Euribor rates are most popular
see `http://www.euribor.org`

After the LIBOR rates were introduced they became quickly popular among banks

Example of LIBOR fixings

External documents

- Euribor tech features (`0010-Euribor-tech-features.pdf`)
- Panel banks (`0020-Euribor-Panel-Banks.pdf`)
- Daily bank quotes (`0030-EURIBOR-panel-Oct-2011.xlsx`)
- Financial Times quotes (`0050-LIBOR-FT-scan-Oct2011.jpg`)
- Historical USD LIBOR (`0070-LIBOR-USD-Rates-History.pdf`)

LIBOR fixings in summer/fall 2008

In summer and fall of 2008, the BBA and other agencies questioned the deposit rates reported by some banks

Since the reported rates are made public, some banks were hiding their financial troubles by reporting a deposit rate that was actually lower than the one they were charged

Some rumors has that this bad practice was used also in September/October 2011

PS: this slided is from last year

The Libor scandal: a timeline

In July 2012 we found out that

2005 Traders start trying to rig Libor, according to the Financial Services Authority (FSA).

2007 Financial crisis aggravates

2008 (October) Allegedly, Barclays notifies the Bank of England, that they are in trouble and need lower interest rates

2008 (November) Internal e-mails at Barclays suggest that BoE is ok with Libor manipulations

2012 The news of Libor rigging is public

Forward discount factor

Given the current risk-free discount factor $D(d)$, we determine the arbitrage-free discount factor at date d_1 for date d_2
By definition of forward discount,

$$C = D^{d_1}(d_2) = \mathsf{PV}^{d_1}(1 \text{ paid at } d_2) \tag{28}$$

so that,

$$D(d_2) = \mathsf{PV}(1 \text{ paid at } d_2) = \mathsf{PV}(C \text{ paid at } d_1) = C\, D(d_1) \tag{29}$$

which implies,

$$D^{d_1}(d_2) D(d_1) = D(d_2) \iff D^{d_1}(d_2) = \frac{D(d_2)}{D(d_1)} \tag{30}$$

Forward interest rates

Given the forward discount factor $D^{d_1}(d_2)$, there is a unique arbitrage-free future interest rate r_{fwd} from date d_1 to date d_2,

$$1 = D^{d_1}(d_2)\,(1 + \tau\, r_{\mathsf{fwd}}) \tag{31}$$

giving

$$r_{\mathsf{fwd}} = \frac{1}{\tau}\left[\frac{D\,(d_1)}{D\,(d_2)} - 1\right] \tag{32}$$

Forward rates can be used to determine the arbitrage-free value of future LIBOR fixings

Forward rate agreements (1/2)

A contract that locks in a rate r_{fra} between dates d_1 and d_2

The net present value (NPV) can be computed as

$$\text{NPV}_{\text{fra}} = D(d_1)\, c_1 + D(d_2)\, c_2 \tag{33}$$

with

$$c_1 = N \qquad \text{and} \qquad c_2 = -N\,(1 + r_{\text{fra}}\hat{\tau}) \tag{34}$$

Note: FRA are usually settled at d_1

Forward rate agreements (2/2)

Theorem:

$$\mathsf{NPV}_{\mathsf{fra}} = N\,D(d_2)\,[\tau\, r_{\mathsf{fwd}}(d_1, d_2) - \hat{\tau}\, r_{\mathsf{fra}}] \tag{35}$$

Proof:

$$\begin{aligned} \mathsf{NPV}_{\mathsf{fra}} &= N\,D(d_1) - N\,(1 + r_{\mathsf{fra}}\hat{\tau})\,D(d_2) \\ &= N\,[D(d_1) - D(d_2) - r_{\mathsf{fra}}\hat{\tau}\,D(d_2)] \\ &= N\left[D(d_2)\frac{D(d_1) - D(d_2)}{D(d_2)} - \hat{\tau}\, r_{\mathsf{fra}} D(d_2)\right] \\ &= N\,[D(d_2)\,\tau\, r_{\mathsf{fwd}}(d_1, d_2) - D(d_2)\,\hat{\tau}\, r_{\mathsf{fra}}]\,, \end{aligned} \tag{36}$$

where, in the last step, we used equation (32)

Questions?

Interest rate futures

- Also known as Eurodollar futures in the USD market
- Traded daily, at maturity fixes as, e.g., the 3-month LIBOR rate
- Quotes as 100-percentage rate
- Position settled daily according to the specific market

External docs on interest-rate futures

- CME Eurodollar conventions (`1010-Futures-Eurodollar.pdf`)
- Eurex interest-rate-futures conventions (`1020-Futures-Eurex-3M.pdf`)
- Interest-rate futures quotes from FT (`1030-Futures-FT-scan10.jpg`)

Interest rate swaps (IRS)

A contract between two parties swapping a series of fixed cash flows for a series of floating-rate cash flows

IRS External material

- IRS diagram from ISDA (`2010-IRS-ISDA-Diagram.pdf`)
- Standard IRS contract from Banca San Paolo (`2020-IRS-SanPaolo.pdf`)
- IRS Examples from StatPro (`IRS-StatPro`)

NPV of an interest-rate swap

The Net Present Value (NPV) is obtaining by netting the positive cash flows with the negative ones.

Since all fixed-rate payments share the same rate:

$$\text{NPV}_{\text{swap}} = N\,A_{\text{floating-leg}} - r_{\text{swap}}\,N\,A_{\text{fixed-leg}} \quad (37)$$

- Fixed payments are made yearly
- Floating payments are made quarterly or semiannually

IRS Floating leg

Each future payment is similar to a forward deposit contract

$$\begin{aligned} A_{\text{floating-leg}} &= D(d_2)\,T(d_2,d_1)\,r_{\text{fwd}}(d_1,d_2) + \ldots + \\ &\quad + D(d_n)\,T(d_{n-1},d_n) r_{\text{fwd}}(d_{n-1},d_n) \end{aligned} \tag{38}$$

assuming 0 fixing days (usually, however is 2).

It can be shown that

$$A_{\text{floating-leg}} \simeq D(d_1) - D(d_n) \tag{39}$$

Proof of equation (39)

Recall,

$$T(d_i, d_j)\, r_{\text{fwd}}(d_i, d_j) = \frac{D(d_i) - D(d_j)}{D(d_j)} \tag{40}$$

Hence,

$$\begin{aligned} A_{\text{floating-leg}} = \\ &= D(d_2)\, T(d_2, d_1)\, r_{\text{fwd}}(d_1, d_2) + \ldots + \\ &\qquad +D(d_n)\, T(d_{n-1}, d_n) r_{\text{fwd}}(d_{n-1}, d_n) = \\ &= D(d_2)\, \frac{D(d_1) - D(d_2)}{D(d_2)} + \ldots + D(d_n)\, \frac{D(d_{n-1}) - D(d_n)}{D(d_n)} = \\ &= [D(d_1) - D(d_2)] + [D(d_2) - D(d_3)] + \ldots + [D(d_{n-1}) - D(d_n)] = \\ &= D(d_1) - D(d_n)\,, \end{aligned} \tag{41}$$

IRS Floating leg with spread

Often an additional spread s is added to the floating payments. In this cases we have

$$\begin{aligned} A_{\text{floating-leg}} &= D(d_2)\,T(d_2,d_1)\ [r_{\text{fwd}}(d_1,d_2)+s]+\ldots+ \\ &\quad +D(d_n)\,T(d_{n-1},d_n)\,[r_{\text{fwd}}(d_{n-1},d_n)+s] \qquad (42) \\ &\simeq D(d_1)-D(d_n)+s\ S_{\text{floating-leg}} \qquad (43) \end{aligned}$$

IRS Fixed leg

The sum of all coupons paid with the same fixed rate r_{swap},

$$A_{\text{fixed-leg}} = D(\hat{d}_2)\,\hat{T}(\hat{d}_1, \hat{d}_2) + \ldots + D(\hat{d}_m)\,\hat{T}(\hat{d}_{m-1}, \hat{d}_m) \qquad (44)$$

To be compared with (39)

Notice that while the term $A_{\text{floating-leg}}$ depends from the discount factor at dates d_1 and d_n, the term $A_{\text{fixed-leg}}$ depends on the discount factor at all dates from inception to maturity

Swap fair rate (1/2)

The fixed rate is chosen "at par" ($\text{NPV}_{\text{swap}} = 0$)

From (37) it is easy to show that

$$0 = A_{\text{floating-leg}} - r_{\text{swap-fair}} A_{\text{fixed-leg}} \tag{45}$$

Hence,

$$r_{\text{swap}-\text{fair}} = \frac{A_{\text{floating-leg}}}{A_{\text{fixed-leg}}} \tag{46}$$

Swap fair rate (2/2)

Taking the reciprocal, i.e. computing $1/r_{\text{swap}-\text{fair}}$,

$$\frac{1}{r_{\text{swap}-\text{fair}}} \simeq \frac{D(\hat{d}_2)\,\hat{T}(\hat{d}_1,\hat{d}_2) + \ldots + D(\hat{d}_m)\,\hat{T}(\hat{d}_{m-1},\hat{d}_m)}{D(d_1) - D(d_n) + s\ S_{\text{floating-leg}}} \tag{47}$$

Since all terms $\hat{T}(\hat{d}_{i-1},\hat{d}_i)$ are almost $T = \hat{T}(\hat{d}_1,\hat{d}_2)$,

$$\frac{1}{T\ r_{\text{swap}-\text{fair}}} \simeq \frac{D(\hat{d}_2) + \ldots + D(\hat{d}_m)}{D(d_1) - D(d_n) + s\ S_{\text{floating-leg}}} \tag{48}$$

i.e. the inverse swap fair rate is directly proportional to the average discount factor over the life of the swap

Quoted IRS fair rates

- Quoted for all major markets
- Financial Times IRS quotes (`2030-IRS-FT-Quotes.jpg`)
- IRS quotes for EUR are known as Eurirs (`2050-EurIRS-Quotes.pdf`)

References

- *Options, future, & other derivatives*, John C. Hull, Prentice Hall (from fourth edition)
- *Interest rate models: theory and practice*, D. Brigo and F. Mercurio, Springer Finance (from first edition)
- *Wikipedia*, `http://en.wikipedia.org/wiki/Interest_rate_swap`

