Vanilla interest rate options

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Summary

- Probability evolution at information arrival
- Brownian motion and option pricing
- Probability measures: physical and risk neutral
- Interest-rate options, Caplets and Black formula
- Caps, Floors, and Collars. Cap/Floor parity
- Bootstrap of volatility term structure cap volatilities
- Options on swaps (Swaptions)
- Historical and implied volatilities
- Volatility smile

Probability evolution at information arrival

- Financial variables are information sensitive
- The probability space itself evolves with time
- Example: Libor leg of a swap

Example of information arrival

Consider the Libor leg of an interest rate swap

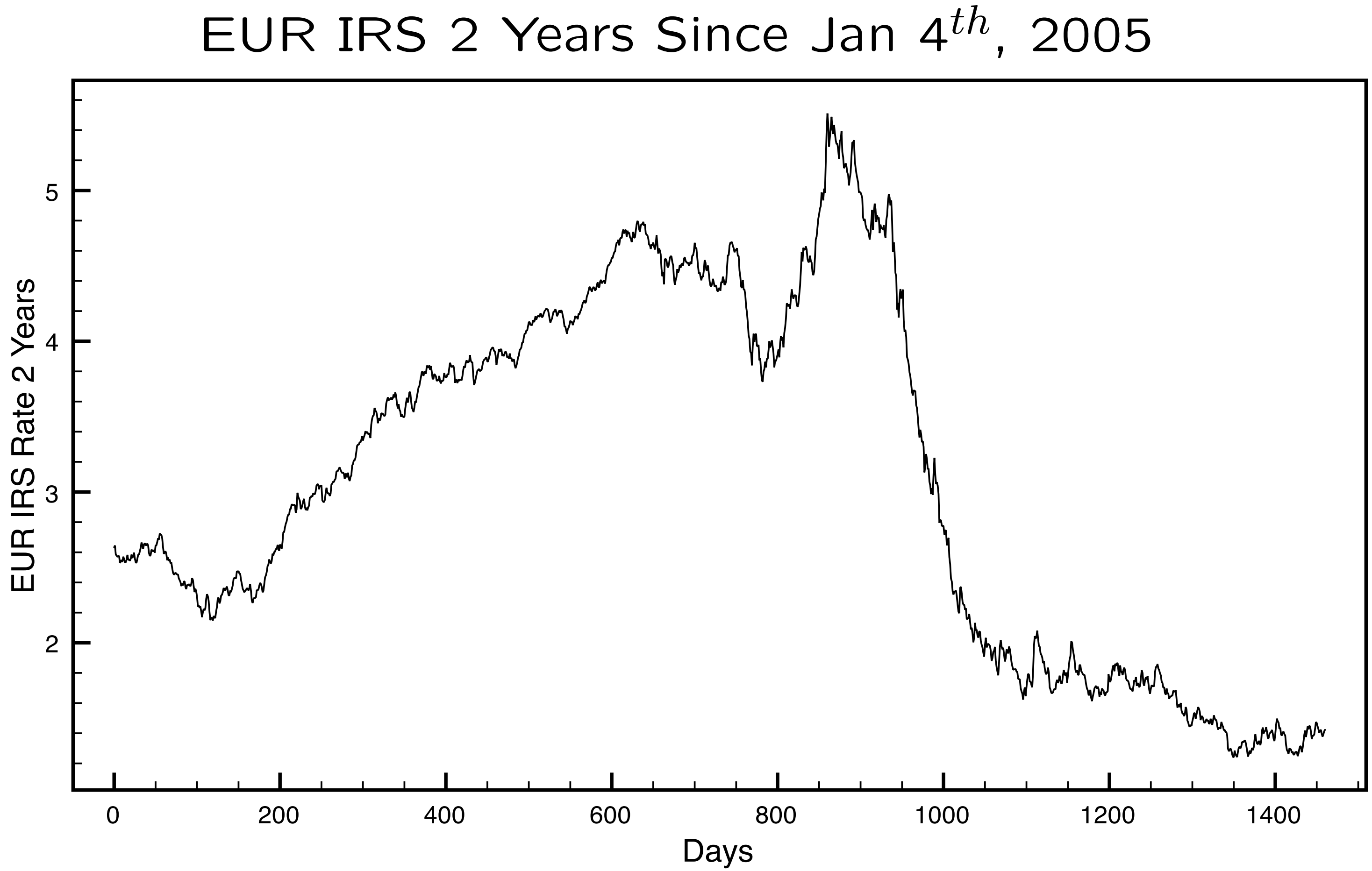
- At the swap issue date only the first payment is known
- During the swap life more and more Libor coupons are discovered
- At the last fixing date all the payments are known

Random variables and stochastic process

Consider a specific rate, e.g., 2-year swap

- The rate at a fixed future date is a random variable
- The rate evolves over time
- The rate at all the future dates constitutes a stochastic process

The simplest (useful) stochastic process is the *Brownian motion*


EUR IRS 2 Years Since Jan 4^{th}, 2005
EUR IRS Rate 2 Years
2
3
4
5
0
200
400
600
800
1000
1200
1400
Days

Important stochastic process: Brownian motion

A stochastic process $t \to B_t$ is a standard Brownian motion when

1. B_0=0
2. $B_t - B_s$ is independent form B_s (for $0 < s < t$)
3. $B_t - B_s$ is normally distributed, precisely as $N(0, t-s)$
4. B_t has continuous trajectories

Probability measures: physical and risk neutral

Explained in details in Derivatives I

- The risk-neutral measure expresses asset values that **must** be agreed by all investors: e.g. up/down probabilities in binomial mono-periodal tree. (Used in derivative pricing.)

- The physical measure is created by the market as a result of the single speculative views of all traders. (Usually this probability is estimated in risk management.)

Questions?

Black-Scholes equation

- The generalization of Brownian motion to a log-normal stochastic process
- Ito's lemma
- No arbitrage arguments and the risk-neutral measure

Resulted in the formulation of the Black-Scholes equation for stock-linked options

The equivalent for the forward price is known as the Black model

Black formula (1/2)

The Black formula is the solution of the Black equation for vanilla options

Consider an option on a variable V_t with forward price F at time T,

$$\text{Payoff}^{\text{call}} = \max(0,\ \text{V}_T\text{-K}) \qquad (1)$$

$$\text{Payoff}^{\text{put}} = \max(0,\ \text{K-V}_T) \qquad (2)$$

Assuming V_T to be log-normal with volatility σ and a payoff deferred to time T^*, the option price is given by the Black formula

Black formula (2/2)

Black formula for a call/put payoff paid at time T^*

$$\mathcal{B}^{\text{call}} = D(T^*)\,[F\,N(d_1) - K\,N(d_2)] \quad (3)$$

$$\mathcal{B}^{\text{put}} = D(T^*)\,[K\,N(-d_1) - F\,N(-d_2)] \quad (4)$$

with

$$d_1 = \frac{\log(F/K)}{\sigma\sqrt{T}} + \sigma\sqrt{T} \quad \text{and} \quad d_2 = \frac{\log(F/K)}{\sigma\sqrt{T}} - \sigma\sqrt{T} \quad (5)$$

and N the cumulative normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\mathrm{d}t \quad (6)$$

Unified Black formula

It is possible to unify the expressions of the Black formula for call and put options by defining call=+1 and put=-1.

The payoff:

$$\text{Payoff}^{\pm} \;=\; \max(0, \pm(V - K)) \tag{7}$$

The Black formula:

$$\mathcal{B}^{\pm} = , \pm D(T^{*}) \left[F\, N(\pm d_1) - K\, N(\pm d_2)\right] \tag{8}$$

QuantLib: Black formula

```
Real blackFormula(optionType, strike, forward,
                  stdDev, discount, displacement) {
  forward = forward + displacement;
  strike = strike + displacement;

  Real d1 = std::log(forward/strike)/stdDev + 0.5*stdDev;
  Real d2 = d1 - stdDev;

  //  optionType is +1 for call options, -1 for put options
  CumulativeNormalDistribution phi;

  Real nd1 = phi(optionType*d1);
  Real nd2 = phi(optionType*d2);

  return discount*optionType*(forward*nd1-strike*nd2);
}
```


Interest-rate (futures) options

An interest-rate vanilla option pays

$$\text{Payoff}^{\pm} \;=\; N\, b \max[0, \pm(F_L - K)] \tag{9}$$

where F_L is the quoted futures rate at option expiry, K is the interest rate strike, N is the notional amount, and b is the basis (e.g. 0.25 for a tenor of 3 months).

The premium of interest-rate options is settled daily (just like interest-rate futures).

Payments are settled at option maturity. Style is American.

Eurodollar Options on Chicago Mercantile Exchange (CME)

	EURODOLLAR OPTIONS
Listed	Eight quarterly options along with two front month serial options
Underlying Contract	Quarterly: Corresponding Quarterly Eurodollar futures Serial: Corresponding Quarterly Eurodollar futures immediately following the serial *Example:* April serial underlying contract is June futures
Minimum Fluctuation	Quoted in IMM Index points One-quarter of one basis point (.0025 = \$6.25) for options when underlying futures is nearest expiring month, and for the first two quarterly months and the first two serial months when the option premium is below five ticks One-half of one basis point (0.005 = \$12.50) for all other contract months
Strike Increment	Strike prices will be listed in intervals of 12.5 basis points (0.125) in a range of 150 basis points above and 150 basis points below the strike closest to the previous day's underlying futures settle price Listed in intervals of 25 basis points (0.25) in a range of 550 basis points above and 550 basis points below the strike closest to the previous day's underlying futures settle price
Last Trading Day	Quarterly: Options trading shall terminate at 11:00 a.m. (London Time) on the second London bank business day before the third Wednesday of the contract month Serial and Mid-Curve: Options trading shall terminate on the Friday immediately preceding the third Wednesday of the contract month. If the foregoing date for termination is an Exchange holiday, options trading shall terminate on the immediately preceding business day
Settlement/Exercise	Options are American Style and are exercised by notifying CME Clearing by 7:00 p.m. CT on the day of exercise. Unexercised options shall expire at 7:00 p.m. CT on the last trading day. In-the-money options that have not been exercised shall be automatically exercised following expiration in the absence of contrary instructions
Trading Hours	Open Outcry: 7:20 a.m. – 2:00 p.m. CT, Monday through Friday CME Globex Electronic Market: 5:00 p.m. – 4:00 p.m. CT, Sunday through Friday

Questions?

Caplets and Floorlets

Given a forward Libor rate from T_1 to T_2 (with year fraction τ), a Caplet (or Floorlet) with strike K has a payoff

$$\text{caplet}^{(+)} = N\,\tau\,\max[0, +(L-K)] \quad (10)$$
$$\text{floorlet}^{(-)} = N\,\tau\,\max[0, -(L-K)] \quad (11)$$

L is the Libor rate at T_1. Usually 3-month Libor is used. Unlike IR options, payments are settled at T_2, or discounted $1/(1+\tau V)$.

- Caplets pay when Libor is *above* the strike.
- Floorlet pay coupons when Libor is *below* the strike

Black formula for Caplets and Floorlets

The market price for Caplets and Floorlets is computed using the Black formula:

$$\text{Optionlet}^{\pm} = \pm N \tau D(T_2) [F N(\pm d_1) - K N(\pm d_2)]$$

Quotes are given in terms of the only free variable: the volatility

Optionlet^{+} is the Caplet; Optionlet^{-} is the Floorlet

Collarlets

A collarlet is a portfolio of a long (positive) Caplet with strike K^{up} and a short (negative) Floorlet with strike K^{dwn} on the same date schedule

$$\begin{aligned} \text{Collarlet}(K^{dwn}, K^{up}) &= \text{Caplet}^{K^{up}} - \text{Floorlet}^{K^{dwn}} = \\ &= N\,\tau \left[\max(0, L - K^{up}) - \max(0, K^{dwn} - L)\right] \\ &= N\,\tau \left[\max(0, L - K^{up}) + \min(0, L - K^{dwn})\right] \end{aligned} \tag{12}$$

Is an insurance on rates staying between K^{dwn} and K^{up}

The strikes K^{dwn} and K^{up} are chosen so that the collar NPV is zero at inception.

Parity formula for Caplets and Floorlets

Consider a collarlet with equal strikes: long a Caplet and a short Floorlet,

$$
\begin{aligned}
&\text{Caplet}^K - \text{Floorlet}^K = \\
&\quad = N\,\tau\,[\max(0, L-K) - \max(0, K-L)] \\
&\quad = N\,\tau\,[\max(0, L-K) + \min(0, L-K)] \\
&\quad = N\,\tau\,(L-K) = \text{FRA}(K, T_1, T_2)
\end{aligned}
$$

Hence, for any volatility and any strike, we have

$$\text{Caplet} = \text{Floorlet} + \text{FRA} \tag{13}$$

All optionlets are considered European style

Interest rate Caps and Floors

Given a Notional and a tenor, usually 3 months, a Cap is a contract that pays all consecutive caplets (but the first), at the same strike, until maturity

Cap maturing in one year

$$\text{Cap} \;=\; \text{Caplet}(3m,6m) + \text{Caplet}(6m,9m) + \text{Caplet}(9m,12m)$$

Similarly, an interest-rate Floor is the sum of forward consecutive Floorlets until maturity

Example: Payoff of a Collar=Cap-Floor

Consider an IRS Libor leg of a payer swap, a long Cap at 4% and a short, i.e. negative, Floor at 2%, all with notional N=400,000 $

Date	**Libor Fix**	**Libor leg ($)**	**Cap ($)**	**-Floor ($)**
T=Today	1.20 %	0	0	0
T + 3m	1.70 %	-1,200	0	0
T + 6m	2.50 %	-1,700	0	-300
T + 9m	4.40 %	-2,500	0	0
T + 12m	—	-4,400	400	0

Cap Floor parity

Recall for Caplets and Floorlets

$$\text{Caplet} - \text{Floorlet} = \text{FRA} \tag{14}$$

Hence consider a portfolio formed by a long Cap and a short Floor with the same strike and maturity

$$\begin{aligned}\text{Cap}(K) \; - \; \text{Floor}(K) &= \text{FRA}(K, 3m, 6m) + \text{FRA}(K, 6m, 9m) \\ &+ \text{FRA}(K, 9m, 12m) + \ldots = \text{ForwardSwap}(3m, K)\end{aligned}$$

The swap is not standard (forward and same leg tenor)

At the money Caps and Floors

$$\text{Cap}(K_{\text{ATM}}) - \text{Floor}(K_{\text{ATM}}) = \text{Forward Swap}(K_{\text{ATM}}) = 0$$

The at-the-money strike brings the forward swap at PAR

Instrument	In the money	Out of the Money
Cap	$K < K_{\text{ATM}}$	$K > K_{\text{ATM}}$
Floor	$K > K_{\text{ATM}}$	$K < K_{\text{ATM}}$

Note that K_{ATM} changes with market conditions and is very close to the current swap rate.

Cap volatility and Spot volatility

- The Cap volatility is the volatility to be used for all Caplets up to maturity so that the sum of the Caplet PVs gives the CAP PV
- A spot volatility is the volatility obtained for each single Caplet so that any CAP PV can be obtained as the sum of the PV of the constituting caplets

The market quotes Cap volatilities

Quotes of Cap/Floor Volatilities

Volatility quotes for at-the-money Caps and Floors for the Euro currency observed on October 15th, 2010.

Years	**Vol (%)**	**Years**	**Vol (%)**
1	36.91	8	32.88
2	46.43	9	31.03
3	43.35	10	29.52
4	42.78	12	27.17
5	40.52	15	25.04
6	37.83	20	23.51
7	35.18	25	—

QuantLib: Caps & Floors

- Constructor: `qlCapFloor`

- Pricing engine: `qlBlackCapFloorEngine`

- At-the-money rate: `qlCapFloorAtmRate`

- Implied volatility: `qlCapFloorImpliedVolatility`

Interpolation of Cap volatilities

- Cap volatilities are interpolated so that the resulting function is smooth
- A popular interpolation function is the *cubic spline*
- A cubic spline is a piecewise-cubic function that matches first and second derivatives at the curve nodes (see Wikipedia for formulas)

Bootstrap of spot-volatility term structure

Write Caps PV as C and Caplets as c. Assume the first caplet vols to be the same as that of the first Cap:

$$\sigma^c_{3m} = \sigma^c_{6m} = \sigma^c_{9m} = \Sigma^C_{1Y} \tag{15}$$

Since

$$\begin{aligned} C(12m) &= c(3m) + c(6m) + c(9m) && (16) \\ C(15m) &= c(3m) + c(6m) + c(9m) + c(12m) && (17) \end{aligned}$$

Compute the caplet vol from the caplet PV

$$\sigma^c_{12m} = \text{ImpliedVol}\,[C(15) - C(12)] \tag{18}$$

Bootstrap of Cap/Floor volatility term structure (1/2)

Quote: 0#EURVBCZ=R

0#EURVBCZ=R REUTERS DEALING RTR IRVOL

EUR	YIELD	VOLATILITY	
TN	0.789	36.771	09:31
1W	0.766	36.771	09:31
1M	0.834	36.771	09:31
2M	0.875	36.771	09:31
3M	1.057	36.771	09:31
6M	1.176	36.771	09:31
9M	1.328	36.771	09:31
1Y	1.317	39.540	09:31
1Y3M	1.349	42.309	09:31
1Y6M	1.376	45.047	09:31
1Y9M	1.436	47.786	09:31
2Y	1.548	44.511	09:31
2Y3M	1.623	41.235	09:31
2Y6M	1.667	38.031	09:31
2Y9M	1.698	37.672	09:31
3Y	1.730	37.342	09:31
3Y3M	1.770	37.005	09:31
3Y6M	1.813	36.690	09:31
3Y9M	1.860	36.363	09:31
4Y	1.912	36.026	09:31
4Y3M	1.962	35.700	09:31
4Y6M	2.012	35.373	09:31
4Y9M	2.061	33.888	09:31
5Y	2.109	32.403	09:31
5Y3M	2.155	30.917	09:31
5Y6M	2.201	29.432	09:31
5Y9M	2.245	27.947	09:31

Bootstrap of Cap/Floor volatility term structure (2/2)

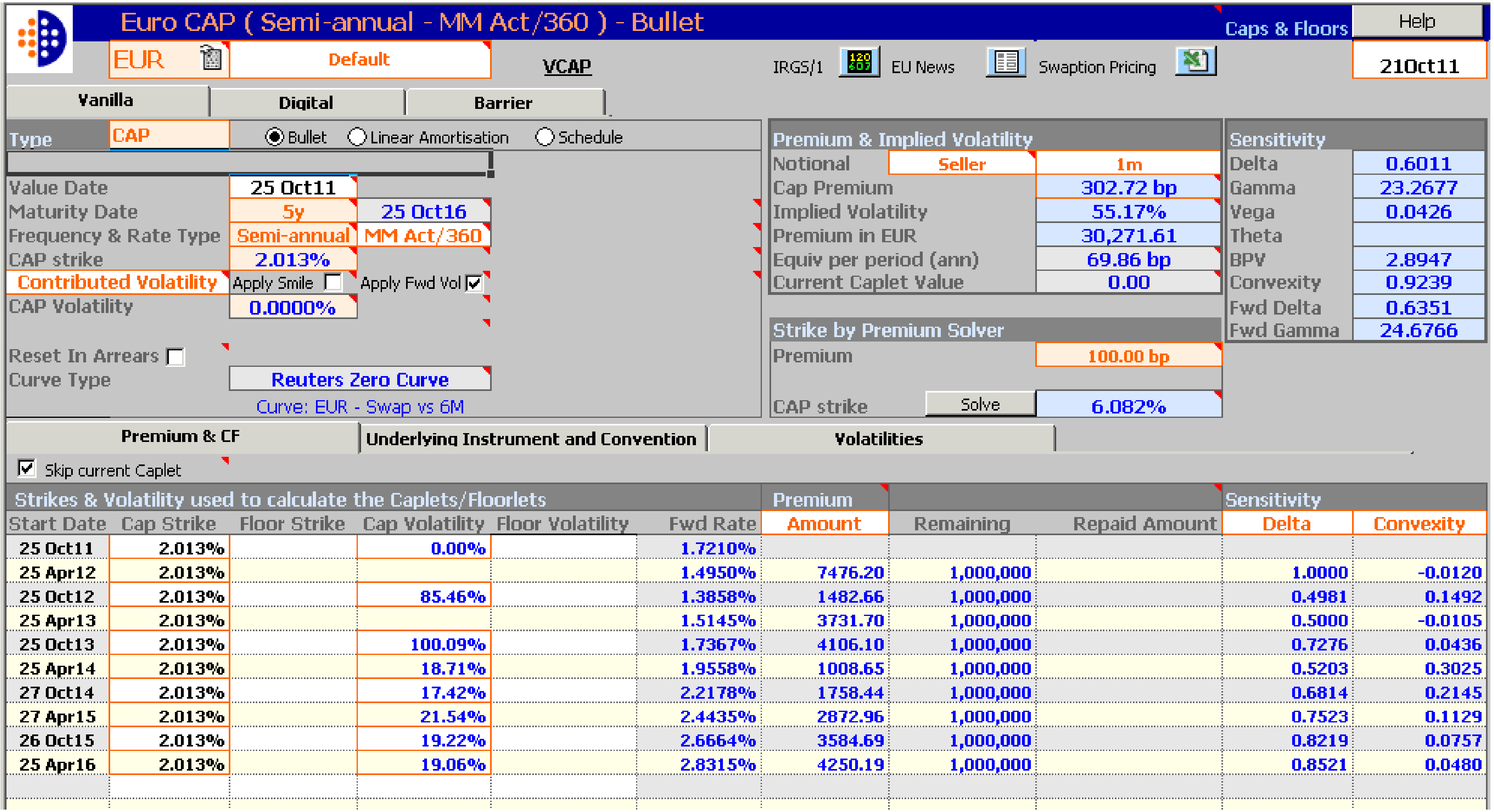
6Y	2.289	26.445	09:31
6Y3M	2.332	24.944	09:31
6Y6M	2.373	23.475	09:31
6Y9M	2.414	23.166	09:31
7Y	2.453	22.868	09:31
7Y3M	2.490	22.569	09:31
7Y6M	2.526	22.277	09:31
7Y9M	2.560	21.982	09:31
8Y	2.593	21.683	09:31
8Y6M	2.655	21.092	09:31
9Y	2.713	20.498	09:31
9Y6M	2.768	19.898	09:31
10Y	2.819	19.591	09:31
11Y	2.918	18.977	09:31
12Y	3.006	18.362	09:31
13Y	3.072	17.746	09:31
14Y	3.131	17.129	09:31
15Y	3.178	16.817	09:31
20Y	3.261	16.817	09:31
25Y	3.163	16.817	09:31
30Y	2.984	16.817	09:31
40Y	2.746	16.817	09:31
50Y	2.686	16.817	09:31

Practical usage of Caps and Floors

Why buy (or sell) Caps and Floors?

- Buy protection on interest-rate hiccups
- Speculate in interest-rate volatilities
- Hedge an interest-rate portfolio
- (other reasons)

Example of computation of Vanilla Cap



Euro CAP (Semi-annual - MM Act/360) - Bullet | Caps & Floors | Help

EUR | Default | VCAP | IRGS/1 | EU News | Swaption Pricing | 21Oct11

Vanilla | Digital | Barrier

Type: CAP — Bullet / Linear Amortisation / Schedule

Value Date	25 Oct11	
Maturity Date	5y	25 Oct16
Frequency & Rate Type	Semi-annual	MM Act/360
CAP strike	2.013%	
Contributed Volatility	Apply Smile ☐	Apply Fwd Vol ☑
CAP Volatility	0.0000%	
Reset In Arrears ☐		
Curve Type	Reuters Zero Curve	
	Curve: EUR - Swap vs 6M	

Premium & Implied Volatility		
Notional	Seller	1m
Cap Premium		302.72 bp
Implied Volatility		55.17%
Premium in EUR		30,271.61
Equiv per period (ann)		69.86 bp
Current Caplet Value		0.00

Sensitivity	
Delta	0.6011
Gamma	23.2677
Vega	0.0426
Theta	
BPV	2.8947
Convexity	0.9239
Fwd Delta	0.6351
Fwd Gamma	24.6766

Strike by Premium Solver		
Premium		100.00 bp
CAP strike	Solve	6.082%

Premium & CF | Underlying Instrument and Convention | Volatilities

☑ Skip current Caplet

Strikes & Volatility used to calculate the Caplets/Floorlets						Premium			Sensitivity	
Start Date	Cap Strike	Floor Strike	Cap Volatility	Floor Volatility	Fwd Rate	Amount	Remaining	Repaid Amount	Delta	Convexity
25 Oct11	2.013%		0.00%		1.7210%					
25 Apr12	2.013%				1.4950%	7476.20	1,000,000		1.0000	-0.0120
25 Oct12	2.013%		85.46%		1.3858%	1482.66	1,000,000		0.4981	0.1492
25 Apr13	2.013%				1.5145%	3731.70	1,000,000		0.5000	-0.0105
25 Oct13	2.013%		100.09%		1.7367%	4106.10	1,000,000		0.7276	0.0436
25 Apr14	2.013%		18.71%		1.9558%	1008.65	1,000,000		0.5203	0.3025
27 Oct14	2.013%		17.42%		2.2178%	1758.44	1,000,000		0.6814	0.2145
27 Apr15	2.013%		21.54%		2.4435%	2872.96	1,000,000		0.7523	0.1129
26 Oct15	2.013%		19.22%		2.6664%	3584.69	1,000,000		0.8219	0.0757
25 Apr16	2.013%		19.06%		2.8315%	4250.19	1,000,000		0.8521	0.0480

Risk exposure of Caps and Floors

What risks do I incur when I buy Caps or Floors?

- Interest-rate risk
- Volatility risk
- Time decay
- If long only the premium paid
- If short potentially unlimited risk (with a small probability)
- Other risks?

- Credit risk used to be disregarded

Questions?

Options on swaps (Swaptions)

A Swaption gives the owner the right (and the seller the obligation) to enter into a swap with at a specified rate

- The premium is paid at the swaption expiry
- Usually swaptions are cash settled (netting payoff and premium)
- Swaptions with physical settlement are considered “exotic”

Swaption Payoff (1/2)

The swaption pays at maturity the NPV of a forward swap:

$$\text{Swap}_K = N\, A_{\text{Libor}} - r_K\, N\, A_{\text{Fixed}} \tag{19}$$

where A_{Libor} is the PV of the Libor leg and

$$\begin{aligned} A_{\text{Fixed}} &= D(M+1y)\,\tau_1 + D(M+2y)\,\tau_2 + \dots \\ &\quad \dots + D(M+10y)\,\tau_{10}\,. \end{aligned} \tag{20}$$

To be compared with the “fair” forward swap

$$\text{Swap}_{\text{fair}} = N\, A_{\text{Libor}} - r_{\text{fair}}\, N\, A_{\text{Fixed}} \tag{21}$$

with $r_{\text{fair}} = A_{\text{Libor}}/A_{\text{Fixed}}$

Swaption Payoff (2/2)

Since the payoff is cash settled we have

$$
\begin{aligned}
\text{Payoff} &= \max\left[\text{Swap}_K - \text{Swap}_{\text{fair}}, 0\right] \\
&= N \max\left[A_{\text{Libor}} - r_K\, A_{\text{Fixed}} - \left(A_{\text{Libor}} - r_{\text{fair}}\, A_{\text{Fixed}}\right), 0\right] \\
&= N \max\left(r_{\text{fair}}\, A_{\text{Fixed}} - r_K\, A_{\text{Fixed}}, 0\right) \\
&= N\, A_{\text{Fixed}} \max\left(r_{\text{fair}} - r_K, 0\right) \qquad (22)
\end{aligned}
$$

since $A_{\text{Fixed}} > 0$.

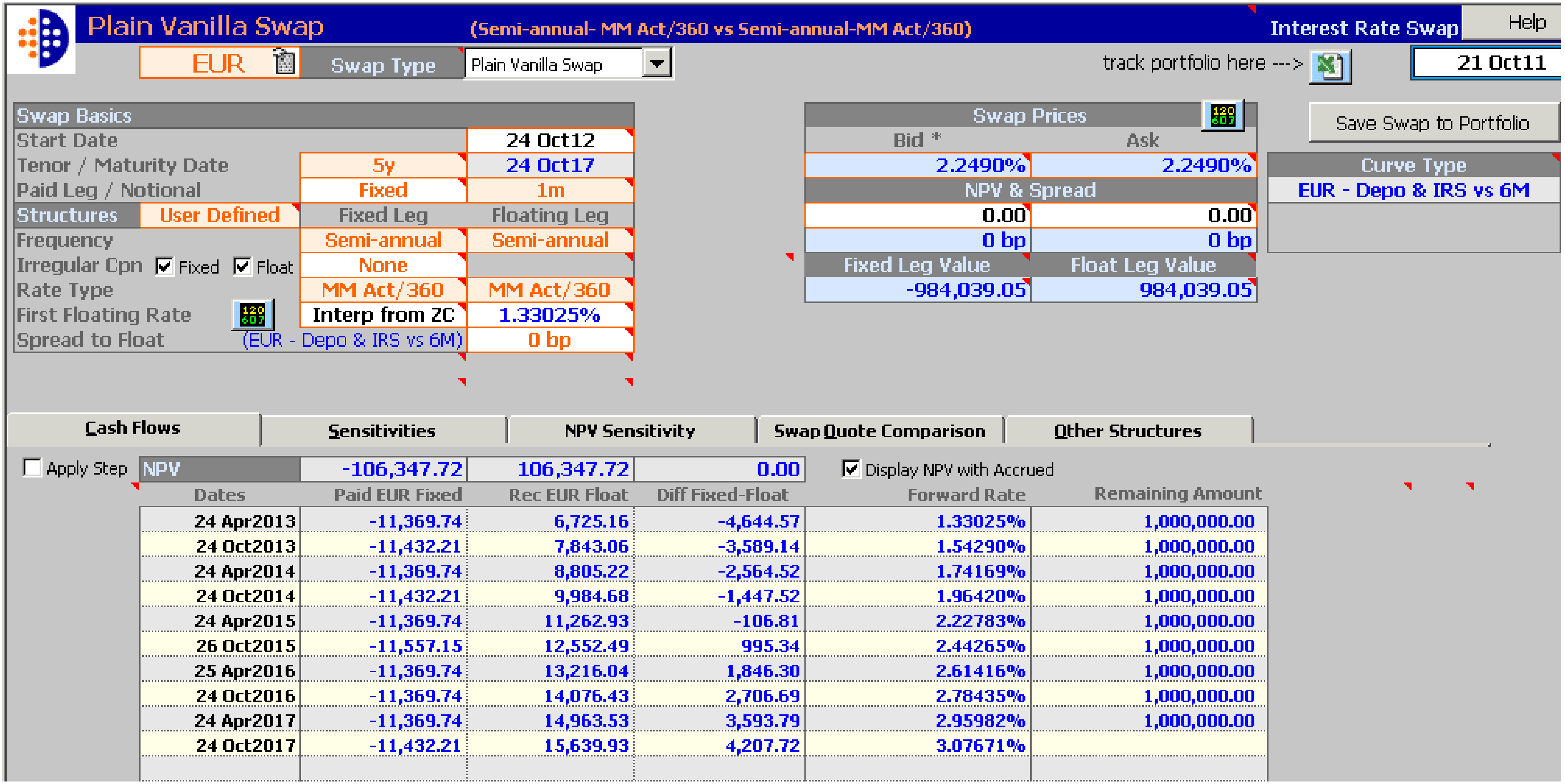
Assuming r_{fair} to be log-normal at the swaption expiry T we can use the Black formula to value the Swaption

Black formula for swaptions

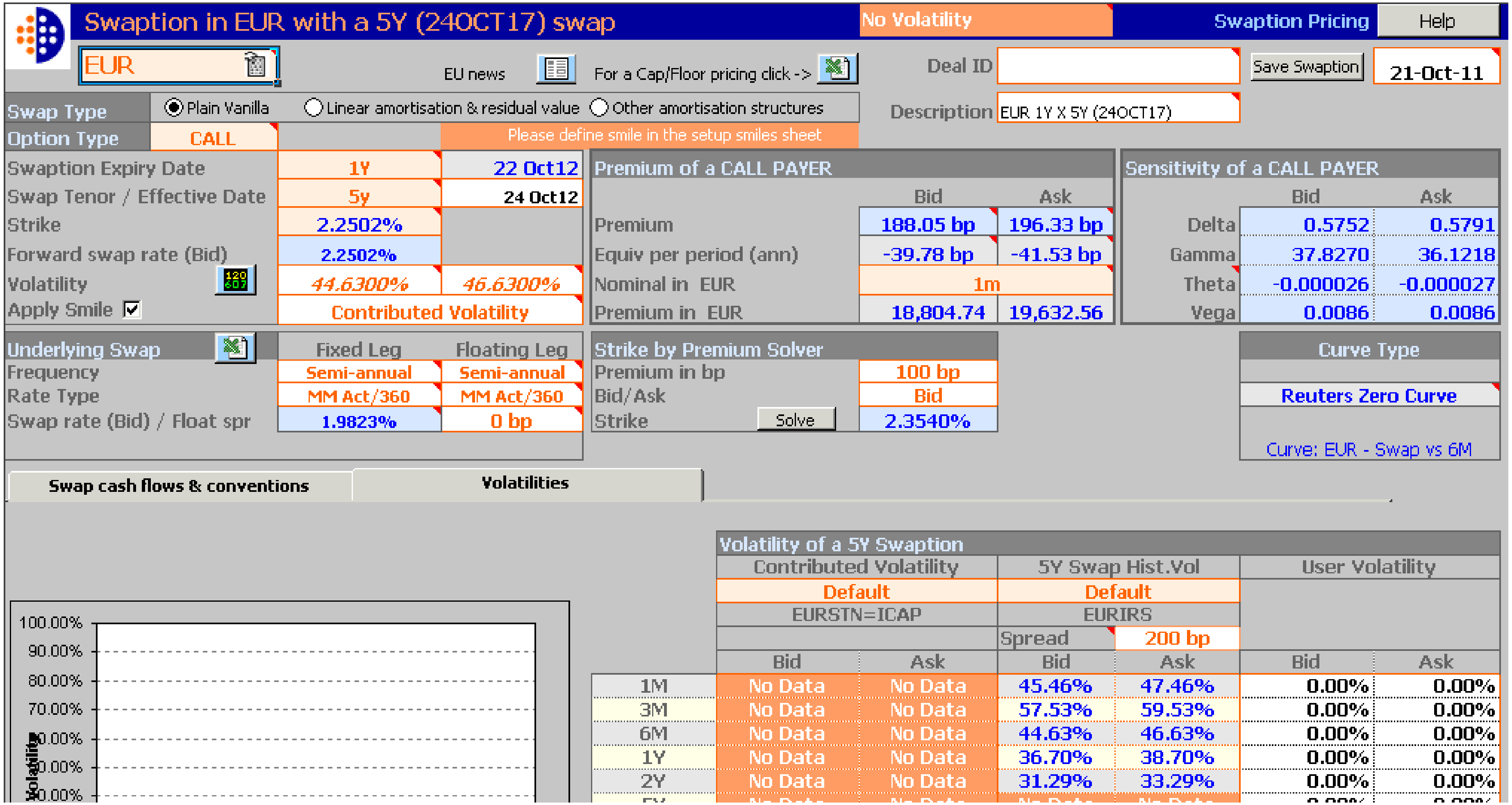
Denoting with + the payer swaption and with - the receiver swaption

$$\text{Swaption}^{\pm} = \pm N A_{\text{Fixed}} \left[r_{\text{fair}} N(\pm d_1) - r_K N(\pm d_2) \right]$$

- Market quotes premiums in terms of volatilities
- Quotes are matrices of option maturities and swap maturities



Dates	Paid EUR Fixed	Rec EUR Float	Diff Fixed-Float	Forward Rate	Remaining Amount
24 Apr2013	-11,369.74	6,725.16	-4,644.57	1.33025%	1,000,000.00
24 Oct2013	-11,432.21	7,843.06	-3,589.14	1.54290%	1,000,000.00
24 Apr2014	-11,369.74	8,805.22	-2,564.52	1.74169%	1,000,000.00
24 Oct2014	-11,432.21	9,984.68	-1,447.52	1.96420%	1,000,000.00
24 Apr2015	-11,369.74	11,262.93	-106.81	2.22783%	1,000,000.00
26 Oct2015	-11,557.15	12,552.49	995.34	2.44265%	1,000,000.00
25 Apr2016	-11,369.74	13,216.04	1,846.30	2.61416%	1,000,000.00
24 Oct2016	-11,369.74	14,076.43	2,706.69	2.78435%	1,000,000.00
24 Apr2017	-11,369.74	14,963.53	3,593.79	2.95982%	1,000,000.00
24 Oct2017	-11,432.21	15,639.93	4,207.72	3.07671%	


Swaption in EUR with a 5Y (24OCT17) swap
No Volatility
Swaption Pricing
Help
EUR
EU news
For a Cap/Floor pricing click ->
Deal ID
Save Swaption
21-Oct-11
Description
EUR 1Y X 5Y (24OCT17)
Swap Type
Plain Vanilla
Linear amortisation & residual value
Other amortisation structures
Option Type
CALL
Please define smile in the setup smiles sheet
Swaption Expiry Date
1Y
22 Oct12
Swap Tenor / Effective Date
5y
24 Oct12
Strike
2.2502%
Forward swap rate (Bid)
2.2502%
Volatility
44.6300%
46.6300%
Apply Smile
Contributed Volatility
Premium of a CALL PAYER
Bid
Ask
Premium
188.05 bp
196.33 bp
Equiv per period (ann)
-39.78 bp
-41.53 bp
Nominal in EUR
1m
Premium in EUR
18,804.74
19,632.56
Sensitivity of a CALL PAYER
Bid
Ask
Delta
0.5752
0.5791
Gamma
37.8270
36.1218
Theta
-0.000026
-0.000027
Vega
0.0086
0.0086
Underlying Swap
Fixed Leg
Floating Leg
Frequency
Semi-annual
Semi-annual
Rate Type
MM Act/360
MM Act/360
Swap rate (Bid) / Float spr
1.9823%
0 bp
Strike by Premium Solver
Premium in bp
100 bp
Bid/Ask
Bid
Strike
Solve
2.3540%
Curve Type
Reuters Zero Curve
Curve: EUR - Swap vs 6M
Swap cash flows & conventions
Volatilities
Volatility of a 5Y Swaption
Contributed Volatility
5Y Swap Hist.Vol
User Volatility
Default
Default
EURSTN=ICAP
EURIRS
Spread
200 bp
Bid
Ask
Bid
Ask
Bid
Ask
1M
No Data
No Data
45.46%
47.46%
0.00%
0.00%
3M
No Data
No Data
57.53%
59.53%
0.00%
0.00%
6M
No Data
No Data
44.63%
46.63%
0.00%
0.00%
1Y
No Data
No Data
36.70%
38.70%
0.00%
0.00%
2Y
No Data
No Data
31.29%
33.29%
0.00%
0.00%
100.00%
90.00%
80.00%
70.00%

Swaption parity

Consider a long Payer Swaption and a short Receiver Swaption with the same strike and maturity, the NPV is given by

$$\begin{aligned} \text{Payoff}^{+} - \text{Payoff}^{-} &= N\,A_{\text{Fixed}}\left[\max\left(r_{\text{fair}} - r_K, 0\right) \right. \qquad (23) \\ &\qquad \left. + \min\left(r_{\text{fair}} - r_K, 0\right)\right] \\ &= N\,A_{\text{Fixed}}\left(r_{\text{fair}} - r_K\right) \qquad (24) \end{aligned}$$

and does not have any optionality. Hence

$$\text{Swaption}^{+} - \text{Swaption}^{-} = N\,A_{\text{Fixed}}\left(r_{\text{fair}} - r_K\right) \qquad (25)$$

The difference between a payer and a receiver swaptions is equal to the NPV of a fixed-fixed swap "fwd-fair" rate against "strike" rate

At the money Swaptions

$$\text{Swaption}^{+} - \text{Swaption}^{-} = N\, A_{\text{Fixed}}\,(r_{\text{fair}} - r_{\text{ATM}}) = 0 \qquad (26)$$

The at-the-money strike r_{ATM} brings the forward swap at par

Swaption	In the money	Out of the Money
Payer	$r_K < r_{\text{ATM}}$	$r_K > r_{\text{ATM}}$
Receiver	$r_K > r_{\text{ATM}}$	$r_K < r_{\text{ATM}}$

Note that r_{ATM} changes with market conditions

Quotes of Swaption ATM Volatilities (1/2)

Option date	1y	2y	3y	4y
2011-11-06	*45.10 %*	*51.20 %*	*55.10 %*	*60.20 %*
2012-01-06	*45.00 %*	*50.60 %*	*54.50 %*	*57.30 %*
2012-04-06	*44.60 %*	*51.50 %*	*54.80 %*	*56.70 %*
2012-10-06	*49.50 %*	*56.20 %*	*58.40 %*	*58.60 %*
2013-10-06	*63.40 %*	*65.80 %*	*63.60 %*	*59.40 %*
2014-10-06	*66.70 %*	*63.10 %*	*58.00 %*	*52.60 %*
2015-10-06	*65.30 %*	*57.40 %*	*50.90 %*	*45.70 %*
2016-10-06	*54.90 %*	*48.20 %*	*43.10 %*	*39.60 %*
2018-10-06	*39.10 %*	*36.20 %*	*34.10 %*	*32.60 %*
2021-10-06	*30.70 %*	*29.90 %*	*28.90 %*	*28.40 %*

www.statpro.com

Quotes of Swaption ATM Volatilities (2/2)

Quote: EURSWPTN1=HVBG

EURSWPTN1=HVBG HVB Swaption LINKED DISPLAYS HVSWAP

	EUR 1Y		EUR 2Y		EUR 3Y		EUR 4Y		
1M	33.0	03:24	35.7	03:24	35.8	03:24	35.2	03:24	1M
3M	36.5	03:24	38.6	03:24	37.4	03:24	36.1	03:24	3M
6M	40.2	03:24	39.6	03:24	37.5	03:24	35.6	03:24	6M
9M	43.2	03:24	40.1	03:24	42.3	10:20	35.7	03:24	9M
1Y	44.7	03:24	40.6	03:24	37.6	03:24	37.1	10:20	1Y
2Y	44.9	03:24	36.7	03:24	33.7	03:24	31.8	03:24	2Y
3Y	37.2	03:24	31.9	03:24	29.7	03:24	28.2	03:24	3Y
4Y	31.1	03:24	27.7	03:24	26.3	03:24	25.2	03:24	4Y
5Y	27.0	03:24	24.6	03:24	23.6	03:24	23.0	03:24	5Y
6Y	24.0	03:24	22.5	03:24	21.1	10:20	21.4	03:24	6Y
7Y	21.5	03:24	20.6	03:24	20.2	03:24	20.0	03:24	7Y
8Y	20.5	03:24	19.7	03:24	19.5	03:24	19.3	03:24	8Y
9Y	19.5	03:24	18.9	03:24	18.8	03:24	18.7	03:24	9Y
10Y	18.5	03:24	18.1	03:24	18.1	03:24	18.2	03:24	10Y
15Y	18.3	03:24	18.2	03:24	18.5	03:24	18.8	03:24	15Y
20Y	20.2	03:24	20.8	03:24	21.5	03:24	22.3	03:24	20Y
30Y	27.0	03:24	26.1	03:24	26.4	03:24	27.0	03:24	30Y

QuantLib: Swaptions

- Constructor: `qlSwaption`

- Pricing engine: `qlBlackSwaptionEngine`

- Implied volatility: `qlSwaptionImpliedVolatility`

Questions?

Historical and implied volatilities

- Historical volatility is computed as the standard deviation of interest-rate daily variation. A single volatility is obtained for all cap/floor maturities. It is **never** used in practice.

- Implied volatility is the **correct** volatility: it is used to provide market quotes for NPV of Cap/Floor and Swaptions

Volatility smile

Generally the market quotes different volatilities for options that are not at the money. Volatilities depend on the strike level.

- Cap/Floor smiles are quoted in volatility matrices

- Swaption smiles are quoted in volatility cubes

Volatility smile for Cap/Floor volatilities

13:30 20OCT10 BHF-BANK GE60220 BHFEURIRG
BHF-BANK Frankfurt / Alexander Titsch-Rivero & Gerhard Langpape / +49-69-718355
EUR Cap&Floor 1Y3M, 2Y+6M / INDICATION / >BHFIROPT<

Exp	Strike	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
1Y	------>	38.90	42.70	45.60	47.90	49.70	51.70	53.00	54.50	55.90
2Y	V	45.80	48.60	50.80	52.70	54.30	55.70	56.70	57.60	58.60
3Y	o	41.60	42.70	44.30	45.80	47.10	45.80	46.00	46.30	46.60
4Y	l	41.00	40.00	40.50	41.40	42.30	40.60	40.60	40.80	40.80
5Y	a	39.50	37.00	36.50	36.80	37.50	36.20	36.30	36.20	36.30
7Y		36.40	32.10	30.20	29.90	30.30	29.70	30.00	30.30	30.60
10Y		33.20	28.00	25.10	24.50	24.90	24.80	25.50	26.10	26.80
1Y	------>	0.01	0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2Y	C	0.23	0.09	0.04	0.02	0.02	0.01	0.01	0.01	0.00
3Y	a	0.87	0.44	0.25	0.16	0.11	0.07	0.04	0.03	0.02
4Y	p	1.69	0.88	0.51	0.33	0.23	0.13	0.09	0.06	0.04
5Y	s	2.78	1.51	0.88	0.56	0.39	0.24	0.17	0.12	0.09
7Y		5.52	3.14	1.82	1.15	0.80	0.52	0.39	0.29	0.23
10Y		10.12	6.03	3.48	2.19	1.54	1.07	0.85	0.69	0.59
1Y	------>	0.56	1.30	2.05	2.80	3.55	4.31	5.06	5.81	6.56
2Y	F	0.95	2.31	3.76	5.23	6.72	8.21	9.70	11.19	12.69
3Y	l	1.37	3.40	5.68	8.05	10.46	12.87	15.32	17.77	20.22
4Y	o	1.78	4.38	7.42	10.65	13.95	17.26	20.63	24.01	27.41
5Y	o	2.11	5.16	8.87	12.88	17.04	21.21	25.47	29.75	34.05
7Y	r	2.64	6.35	11.12	16.54	22.28	28.09	34.05	40.04	46.07
10Y	s	3.34	7.75	13.70	20.92	28.77	36.80	45.08	53.42	61.82

References

- *Options, future, & other derivatives*, John C. Hull, Prentice Hall (from fourth edition)

- *Interest rate models: theory and practice*, D. Brigo and F. Mercurio, Springer (from first edition)

