Exotic interest-rate options

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Lecture Summary

- Exotic caps, floors, and swaptions
- Swaps with exotic floating-rate legs
- No-arbitrage methods (commodity example)
- Numeraires and pricing formulas
- stochastic differential equations (SDEs)
- Partial differential equations (PDEs)
- Feynman-Kac formula
- Numerical methods: analytical approximations

Common exotic interest-rate options

There are all sorts of exotic interest-rate options.

The least exotic, i.e. commonly traded, types are

- Caps and floors with a digital payoff
- Caps and floors with barriers
- Bermuda and American Swaptions

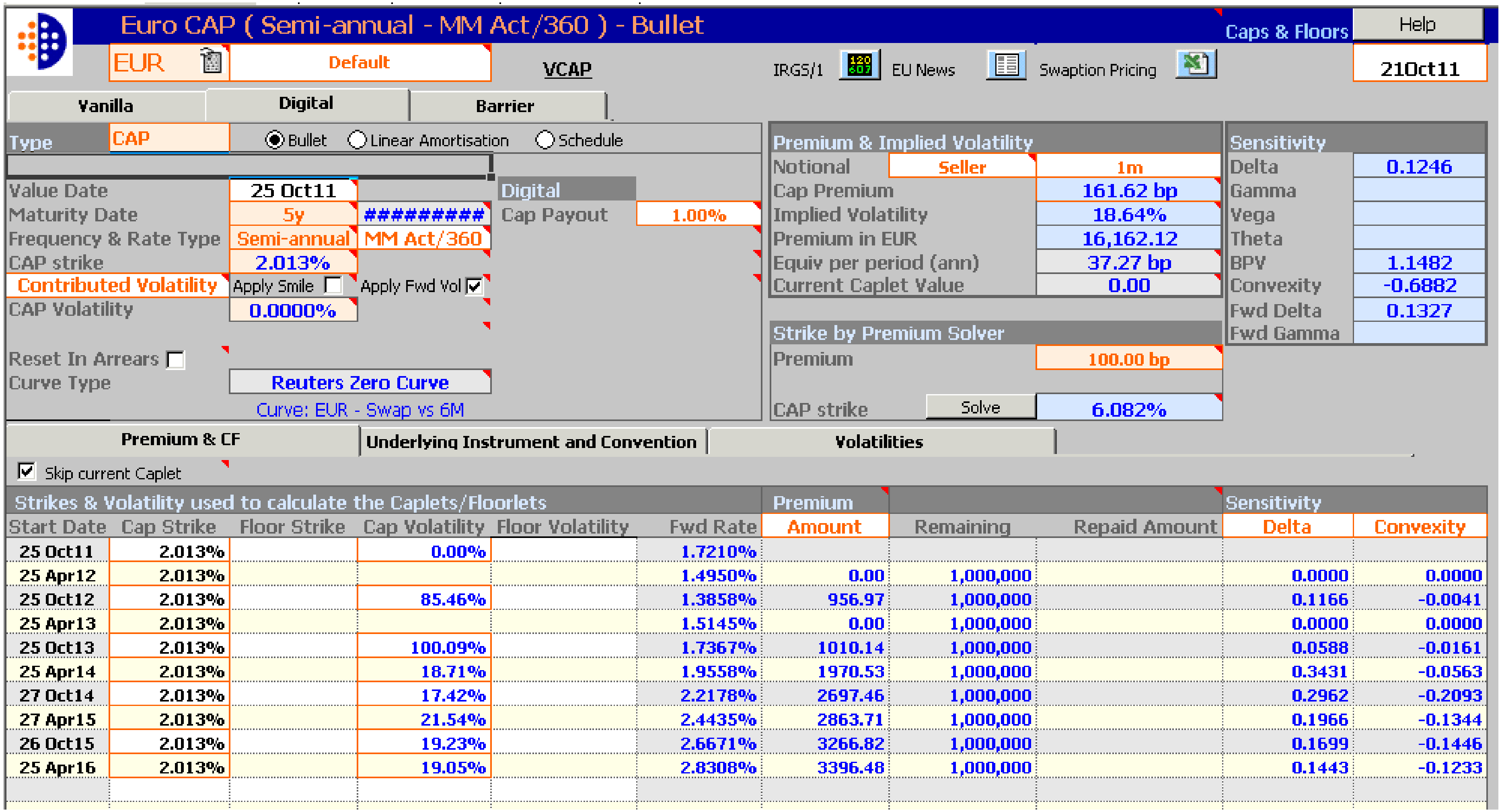
Caps and floors with a digital payoff

A vanilla Cap has caplets with payoff

$$\text{Caplet}_V^{\text{Payoff}} = N\,T(6m, 9m)\,\text{max}\left[(\text{3m-Libor} - K), 0\right] \tag{1}$$

A *digital* Cap has caplets with payoff

$$\text{Caplet}_D^{\text{Payoff}} = \begin{cases} N\,T(6m, 9m)\,R & \text{for} \quad \text{3m-Libor} \geq K \\ \\ 0 & \text{for} \quad \text{3m-Libor} < K \end{cases} \tag{2}$$



Start Date	Cap Strike	Floor Strike	Cap Volatility	Floor Volatility	Fwd Rate	Amount	Remaining	Repaid Amount	Delta	Convexity
25 Oct11	2.013%		0.00%		1.7210%					
25 Apr12	2.013%				1.4950%	0.00	1,000,000		0.0000	0.0000
25 Oct12	2.013%		85.46%		1.3858%	956.97	1,000,000		0.1166	-0.0041
25 Apr13	2.013%				1.5145%	0.00	1,000,000		0.0000	0.0000
25 Oct13	2.013%		100.09%		1.7367%	1010.14	1,000,000		0.0588	-0.0161
25 Apr14	2.013%		18.71%		1.9558%	1970.53	1,000,000		0.3431	-0.0563
27 Oct14	2.013%		17.42%		2.2178%	2697.46	1,000,000		0.2962	-0.2093
27 Apr15	2.013%		21.54%		2.4435%	2863.71	1,000,000		0.1966	-0.1344
26 Oct15	2.013%		19.23%		2.6671%	3266.82	1,000,000		0.1699	-0.1446
25 Apr16	2.013%		19.05%		2.8308%	3396.48	1,000,000		0.1443	-0.1233

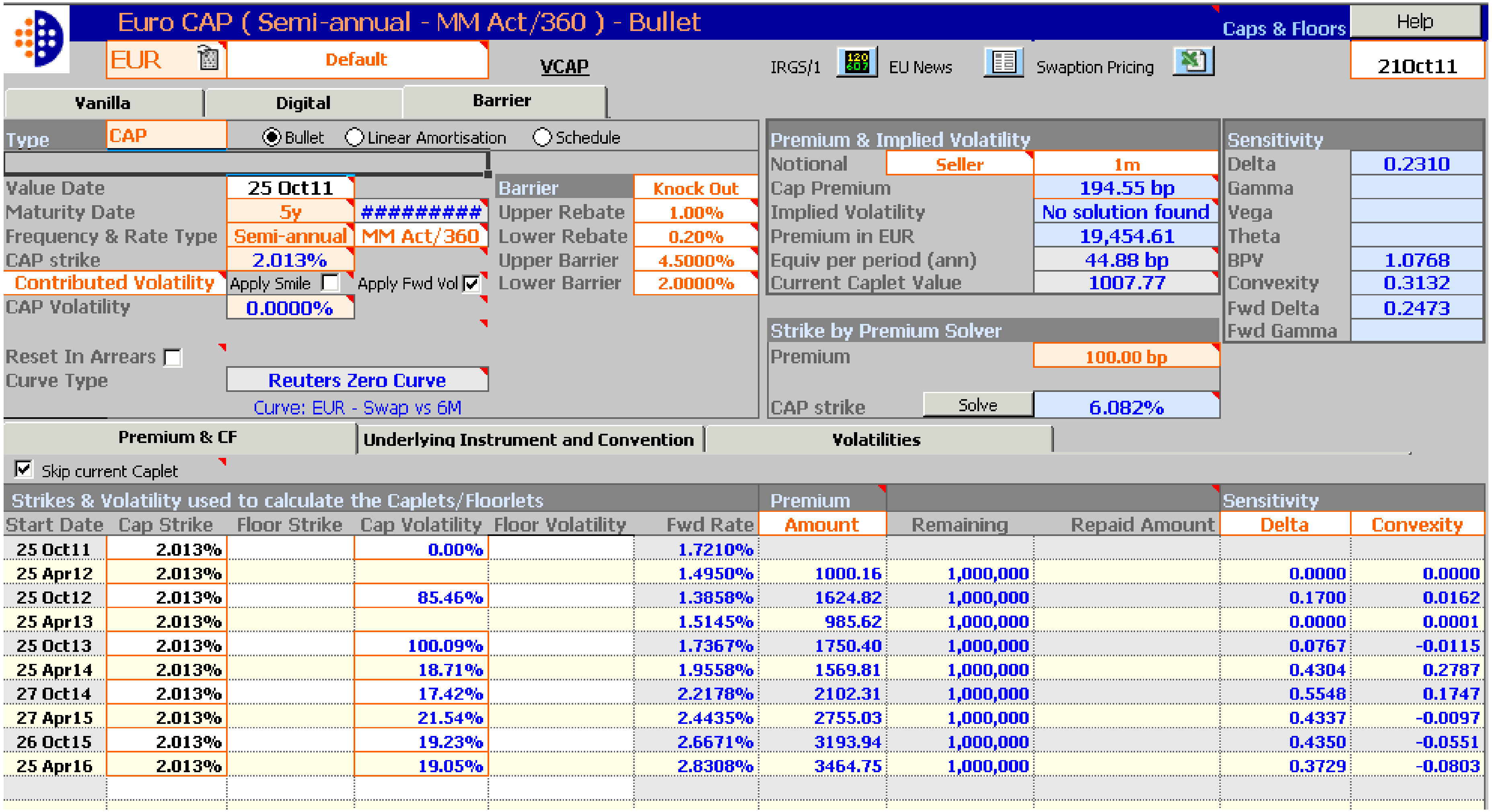
Caps and floors with barriers

A barrier Cap has caplets with payoff

$$\text{Caplet}_V^{\text{Payoff}} = N\,T(6m, 9m)\,\max\left[(\text{3m-Libor} - K), 0\right] \qquad (3)$$

- Knock-in: paid if 3m-Libor **touches** a barrier $r_{\text{knock-in}}$
- Knock-out: paid if 3m-Libor **does not** reach $r_{\text{knock-out}}$

Discrete barriers are checked at certain given barrier dates
Continuous barriers are checked on each daily Libor fixing
Digital payoffs are also available. A *rebate* is paid upon knock out.



Euro CAP (Semi-annual - MM Act/360) - Bullet

Caps & Floors | Help

EUR | Default | VCAP | IRGS/1 | EU News | Swaption Pricing | 21Oct11

Vanilla | Digital | Barrier

Type: CAP | Bullet | Linear Amortisation | Schedule

Value Date	25 Oct11	
Maturity Date	5y	#########
Frequency & Rate Type	Semi-annual	MM Act/360
CAP strike	2.013%	
Contributed Volatility	Apply Smile ☐	Apply Fwd Vol ☑
CAP Volatility	0.0000%	

Reset In Arrears ☐

Curve Type: Reuters Zero Curve

Curve: EUR - Swap vs 6M

Barrier	Knock Out
Upper Rebate	1.00%
Lower Rebate	0.20%
Upper Barrier	4.5000%
Lower Barrier	2.0000%

Premium & Implied Volatility		
Notional	Seller	1m
Cap Premium		194.55 bp
Implied Volatility		No solution found
Premium in EUR		19,454.61
Equiv per period (ann)		44.88 bp
Current Caplet Value		1007.77

Sensitivity	
Delta	0.2310
Gamma	
Vega	
Theta	
BPV	1.0768
Convexity	0.3132
Fwd Delta	0.2473
Fwd Gamma	

Strike by Premium Solver

Premium: 100.00 bp

CAP strike: Solve | 6.082%

Premium & CF | Underlying Instrument and Convention | Volatilities

☑ Skip current Caplet

Strikes & Volatility used to calculate the Caplets/Floorlets

Start Date	Cap Strike	Floor Strike	Cap Volatility	Floor Volatility	Fwd Rate	Premium Amount	Remaining	Repaid Amount	Sensitivity Delta	Convexity
25 Oct11	2.013%		0.00%		1.7210%					
25 Apr12	2.013%				1.4950%	1000.16	1,000,000		0.0000	0.0000
25 Oct12	2.013%		85.46%		1.3858%	1624.82	1,000,000		0.1700	0.0162
25 Apr13	2.013%				1.5145%	985.62	1,000,000		0.0000	0.0001
25 Oct13	2.013%		100.09%		1.7367%	1750.40	1,000,000		0.0767	-0.0115
25 Apr14	2.013%		18.71%		1.9558%	1569.81	1,000,000		0.4304	0.2787
27 Oct14	2.013%		17.42%		2.2178%	2102.31	1,000,000		0.5548	0.1747
27 Apr15	2.013%		21.54%		2.4435%	2755.03	1,000,000		0.4337	-0.0097
26 Oct15	2.013%		19.23%		2.6671%	3193.94	1,000,000		0.4350	-0.0551
25 Apr16	2.013%		19.05%		2.8308%	3464.75	1,000,000		0.3729	-0.0803

Bermuda and American Swaptions

Very similar to standard Swaptions (known as European swaptions) can be exercised at other dates

- Bermuda swaptions: exercised dates are discrete (typically with the same tenor as one of the legs)
- American Swaption can be exercised at any time

Both are found in two different types of payoffs: *co-terminal swap* and *constant-maturity swap*

Questions?

More exotic interest-rate options

- Caps and floors on Constant-Maturity-Swaps rates (CMS)
- Look-back options
- TARN (Target Accumulator Redemption Note) swap legs
- Spread options

Usually these options are *gift-wrapped* within a swaps paying a fixed or a Libor rate

Exotic swap legs

- Constant-maturity swaps (CMS)
- Spread options
- Ratchet swap
- Range accruals
- Look-back options
- TARN (Target Accumulator Redemption Note) swap legs

Constant-maturity swaps (CMS)

A swap with a standard fixed-rate leg and a floating-rate leg that pays at times T_i, with $i = 1, \ldots, n$ cash flows given by

$$C_i = N \, \tau_{i-1} \, \mathrm{CMS}_M(T_{i-1}) \tag{4}$$

where

- τ_i is the year fraction between dates T_i and T_{i+1}
- $\mathrm{CMS}_M(T_i)$ is the fair rate, observed at time T_i, of a vanilla swap (fixed leg against Libor leg) with maturity date $T_i + M$

Constant-maturity swaps usually have Caps or Floors on the swap rate

Swaps with spread options

A swap with a standard fixed-rate leg and a floating-rate leg that pays a cash flow given by the difference of two swap rates.

For example the floating leg fixing at time T_{i-1}, and paying at time T_i, payments are given by

$$C_i = N\,\tau_{i-1}\,\max\left[r_{\text{floor}}, \text{CMS}_{10Y}(T_{i-1}) - \text{CMS}_{2Y}(T_{i-1})\right]$$

- We are betting on the swap curve to steepen
- Usually a multi-factor model is needed to price this instrument

Swap with ratchet options

Also known as *cliquet* option.
A swap with a standard fixed-rate leg and a floating-rate leg that pays at times $T_2, \ldots, T_n$ Caplet-like coupons,

$$C_i = N\, \tau_{i-1} \max\left(0, L_{i-1} - K_{i-1}\right) \tag{5}$$

where L_i is the Libor rate observed at time T_i, and K_i is the strike satisfying

$$K_{i-1} = \max\left(K_{i-2}, L_{i-2}\right) \quad \text{for } i = 3, \ldots \tag{6}$$

Note: coupon rates are not decreasing. Usually $K_1 = Fwd_{12}$

Swaps with range accruals

A swap with a floating-rate leg paying coupons

$$C_i = N\,\tau_{i-1}\,\langle r \rangle \tag{7}$$

where the *average rate* $\langle r \rangle$ is given by

$$\langle r \rangle = \frac{1}{K} \sum_{k \in [T_{i-1}, T_i)} f[L(t_k)] \tag{8}$$

K business dates between T_{i-1} and T_1,
f being the *range* function of Libor rate

$$f(L) = \begin{cases} r_{\text{in}} & \text{for } L_{\text{min}} \le L \le L_{\text{max}} \\ r_{\text{out}} & \text{otherwise} \end{cases} \tag{9}$$

L_{min} minimum Libor rate and L_{max} maximum Libor rate

Look-back options

A swap with a floating-rate leg paying coupons on the maximum Libor fixing over a certain past period

$$C_i = N \ \tau_{i-1} \ r_{\text{max}}(T_{i-1}, T_i) \tag{10}$$

where

$$r_{\text{max}} = \text{max}_{k \in [T_{i-2}, T_{i-1})}[L(t_k)] \tag{11}$$

- We always receive the best Libor fixing in a range
- Sometimes look-back periods extend further back than one coupon

TARN (Target Accumulator Redemption Note) swap legs

A swap with a floating-rate leg paying coupons up to a maximum accumulated rate r_{max}

$$C_i = N\ \tau_{i-1}\, c_i \tag{12}$$

where

$$\begin{aligned}
c_1 &= \min\left[r_{\text{max}},\, L_0\right] \\
c_2 &= \min\left[r_{\text{max}} - c_1, L_1\right] \\
c_3 &= \min\left[r_{\text{max}} - c_1 - c_2, L_2\right] \\
\ldots &
\end{aligned}$$

Coupon rates are subtracted to the maximum rate until a zero rate is reached

Questions?

Crash course on commodity contracts

(Example of no-arbitrage methods)

- Introduction to commodities
- The forward-spot and the forward-forward relationship
- Convenience yields
- Example of commodity derivatives: The forward contract and the futures spread

Exposure to commodity prices

If you are managing an hedge fund and want some exposure on the price of live cattle what can you do?

A. Become a cowboy overnight and buy some live cattle

B. Enter in a derivative contract that gives you an exposure to the live-cattle price (e.g. a futures contract)

Commodity futures contracts

- Commodity futures are among the oldest financial instrument traded on any trading floor
- Futures on the cotton price, for example, have traded on the market for longer than a century
- Refer to specialized literature for more details
- In this talk we describe how to generate simulated spot prices and convenience-yield curves (to be used in the computation of risk figures)

Copper, Corn, and WTI Oil

We consider three samples commodities

- Copper (a.k.a Dr. Copper): non-ferrous industrial metal quoted, in \$ per ton, on the London Metal Exchange (LME)

- Corn: an agriculture commodity quoted, in \$-cents per bushel (27,216 kg), on the Chicago Board of Trade (CBoT)

- West-Texas-Intermediate Oil: an energy commodity quoted, in \$ per barrel (158.987 liters), on the New York Mercantile Exchange (NYMEX)

Commodity futures contracts

A contract that allows the delivery of a commodity at a certain future date (however settled daily according to the close price).

E.g. on 2011-11-01 we observed the following market quotes

Commodity	**Maturity**	**Date**	**Price**
copper	2012G	2012-02-24	7993.00
copper	2012H	2012-03-27	7995.00
copper	2012J	2012-04-25	7995.50

Commodity forward contracts

A **commodity forward contract** is struck between two parties that agree to buy/sell a given commodity at a future date at a pre-determined price (the strike price).

Settlement could be physical or cash. In case of cash settlement the payoff P at the maturity date T (for the long side) is given by the difference between the value of the underlying commodity spot price $S(T)$ and a strike price K.

For W lots we have,

$$P = W \cdot \Big[S(T) - K\Big] \tag{13}$$

Arbitrage-free strategies

An arbitrage-free strategy is ...

A series of physical or financial transactions that starting with a portfolio with a zero value end up with a risk-less portfolio.

The no-arbitrage assumption states that the final portfolio value is zero with 100% probability (otherwise we could buy the cheaper part and sell the dearer one making a risk-free profit with some probability).

Spot-forward relationship (1/2)

For a *storable commodity*. Consider a forward contract on a certain storable commodity (no costs no benefits) asset and set up the following strategy:

Today borrow an amount S of currency, exactly enough to buy the spot asset (since T is a short maturity the borrowing can be made at reasonable, i.e. risk-free like, interest rates) and enter into a short forward contract to sell the asset at a future date T at a price f; then store the asset until the date T; when the date T comes, enforce the contract and sell the asset for a price f_T, finally, payback the loan with interests.

Spot-forward relationship (2/2)

The strategy can be summarized in the following table

Date	**Description**	**Cash flows**	**Asset exch.**
$t = 0$	borrow an amount S of cash purchase the asset at spot price enter into a short forward at T	$+S$ $-S$ 0	+ asset
$0 < t < T$	store the asset		
$t = T$	use the asset to service the forward repay debt with interests	f_T $-S[1 + R(T)\,T]$	- asset

Because of no arbitrage we have

$$f_T = S\Big[1 + R(T)\,T\Big] \tag{14}$$

Adding costs and benefits

In the case of a commodity, for a dividend-paying stock, for a bond, and for some other asset types, in general there are costs and/or benefits associated in holding the asset,

$$f_T = S[1 + R(T)\,T] + \mathrm{Cos}(T) - \mathrm{Ben}(T) \tag{15}$$

Assuming costs and benefits to be proportional to the asset price, using simple compounding,

$$f_T = S\Big[1 + R\,T\Big] + S\Big[Y_{\mathrm{Cos}}\,T - Y_{\mathrm{Ben}}\,T\Big] \tag{16}$$

Convenience yield

We can define the convenience yield as the difference between benefits and costs,

$$y = Y_{\mathsf{Ben}} - Y_{\mathsf{Cos}} \tag{17}$$

so that

$$f_T = S\Big[1 + R\,T - Y\,T\Big] = S\,\frac{e^{-y\,T}}{e^{-r\,T}} \tag{18}$$

where we used continuous compounding. Using the discount factor we have,

$$f_T = \frac{S}{D(T)} e^{-y\,T} \tag{19}$$

Note: the **convenience yield** y usually depends on the maturity T.

Forward-forward relationship

Strategy: at time $t = 0$ enter into a long forward contract to buy the asset at T_1 for f_1 units of currency, at the same time enter into a short contract to sell the asset at date T_2 for a price f_2; at date $t = T_1$ use the long forward contract to buy the asset for a price of f_1 financing the purchase by borrowing the money from the market; at a later date T_2 sell the commodity for f_2

From arbitrage-free assumption we have

$$f_2 = f_1 \, \frac{D(T_1)}{D(T_2)} \, e^{-y_{12}\,(T_2 - T_1)} \tag{20}$$

where y_{12} is the **forward convenience yield** between T_1 and T_2.

General spot-forward relationship

The forward-forward relationship between T_2 and T_3 is

$$f_3 = f_2 \, \frac{D(T_2)}{D(T_3)} \, e^{-y_{23}\,(T_3 - T_2)} \tag{21}$$

Chain linking the spot-forward formula and multiple forward-forward relationships we can write

$$f_j = \frac{S}{D(T_j)} e^{-y_j\,T_j} \tag{22}$$

where

$$y_j\,T_j = y_1\,T_1 + y_{12}\,(T_2 - T_1) + y_{23}\,(T_3 - T_2) + \ldots + y_{ij}\,(T_j - T_i)$$

Forward-futures relationship

The main difference between a futures contract and a forward contract is that the former must be settled daily while the latter is settled at the contract maturity. It can be shown that we have,

$$F^T = f^T + D(T)\,\sigma_S\,\sigma_r\,\rho_{r-S} \tag{23}$$

- σ_s is the asset-price volatility
- σ_r is the interest-rate volatility
- ρ_{S-r} is the correlation between the asset price and the money-market account.

Convenience yields from futures quotes

We assume $|\sigma_S \, \sigma_r \, \rho_{r-S}| \ll 1$ so that

$$F^T \simeq f^T \tag{24}$$

Consider n futures contracts with maturities $T_1, T_2, \ldots T_n$, having, respectively, quotes F_1 $F_1, F_2, \ldots F_n$ at some reference date.

From equation (22) and (24) we have

$$F_i = \frac{S}{D(T_i)} e^{-y_i T_i}, \qquad \text{for } i = 1, \ldots, n \tag{25}$$

compute the convenience yields y_i's as

$$y_i = \frac{1}{T_i} \log \left(\frac{S}{D(T_i) F_i} \right) \tag{26}$$

Yield on a commodity (1/2)

What is exactly the meaning of the convenience yield? Recall that

$$f_T = S \, \frac{1 + r\,T}{1 + y\,T} \tag{27}$$

and suppose we owe a commodity that is worth S today. Assume no credit risk and follow the strategy:

- Today sell the commodity; invest the money in risk-free deposit; enter in a forward contract to buy $W = 1 + yT$ lots at T
- At T recover the money and interests from the depo, and use the money to purchase the commodity at the forward-contract price

Yield on a commodity (2/2)

In summary, starting with 1 lot of the asset

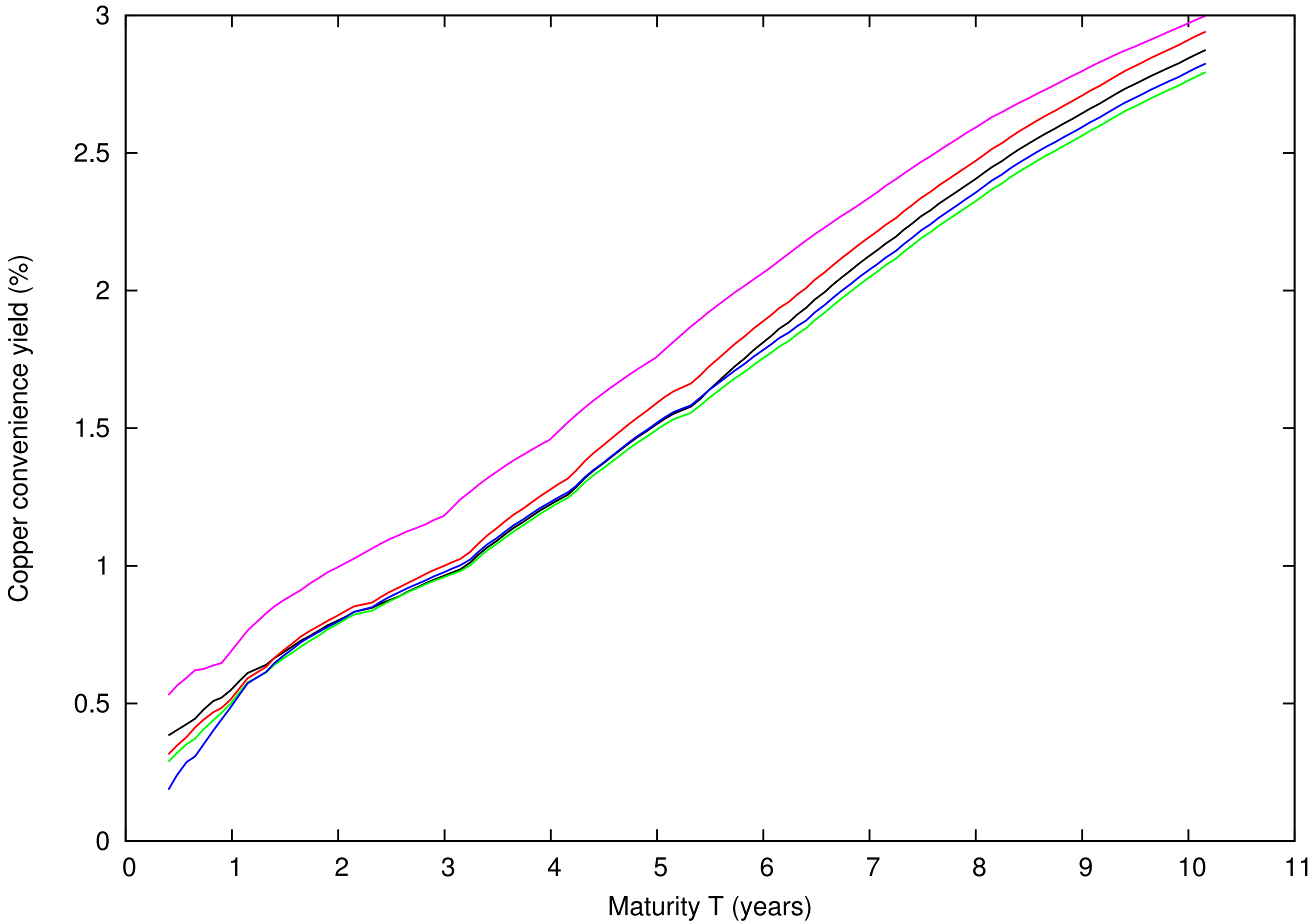
Date	Description	Cash flows	Asset exch.
$t = 0$	sell the asset at spot price S	$+S$	-1 $\cdot$ asset
	invest the money in a deposit	-S	
	enter into W long fwd contracts	0	
$0 < t < T$	do nothing		
$t = T$	redeem the deposit	$+S[1 + r\,T]$	
	use money to service the contract	$-S[1 + r\,T]$	$+W \cdot$ asset

with $W = 1 + y\,T$. The contract payoff at redemption is given by

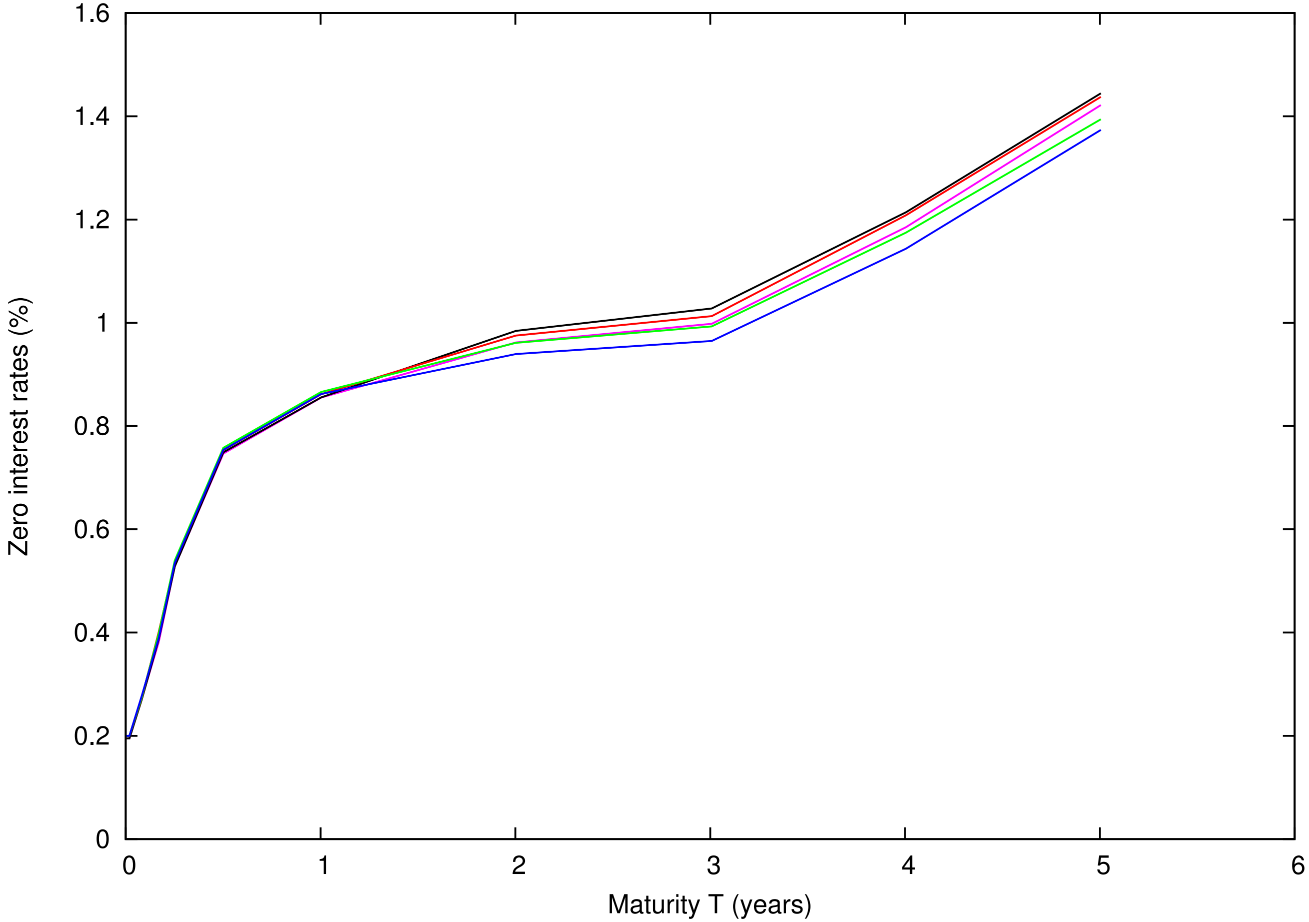
$$W \cdot f_T = (1 + y\,T) \cdot \left(S \, \frac{1 + r\,T}{1 + y\,T} \right) = S\,(1 + r\,T)$$

The commodity provided an interest $y\,T$!

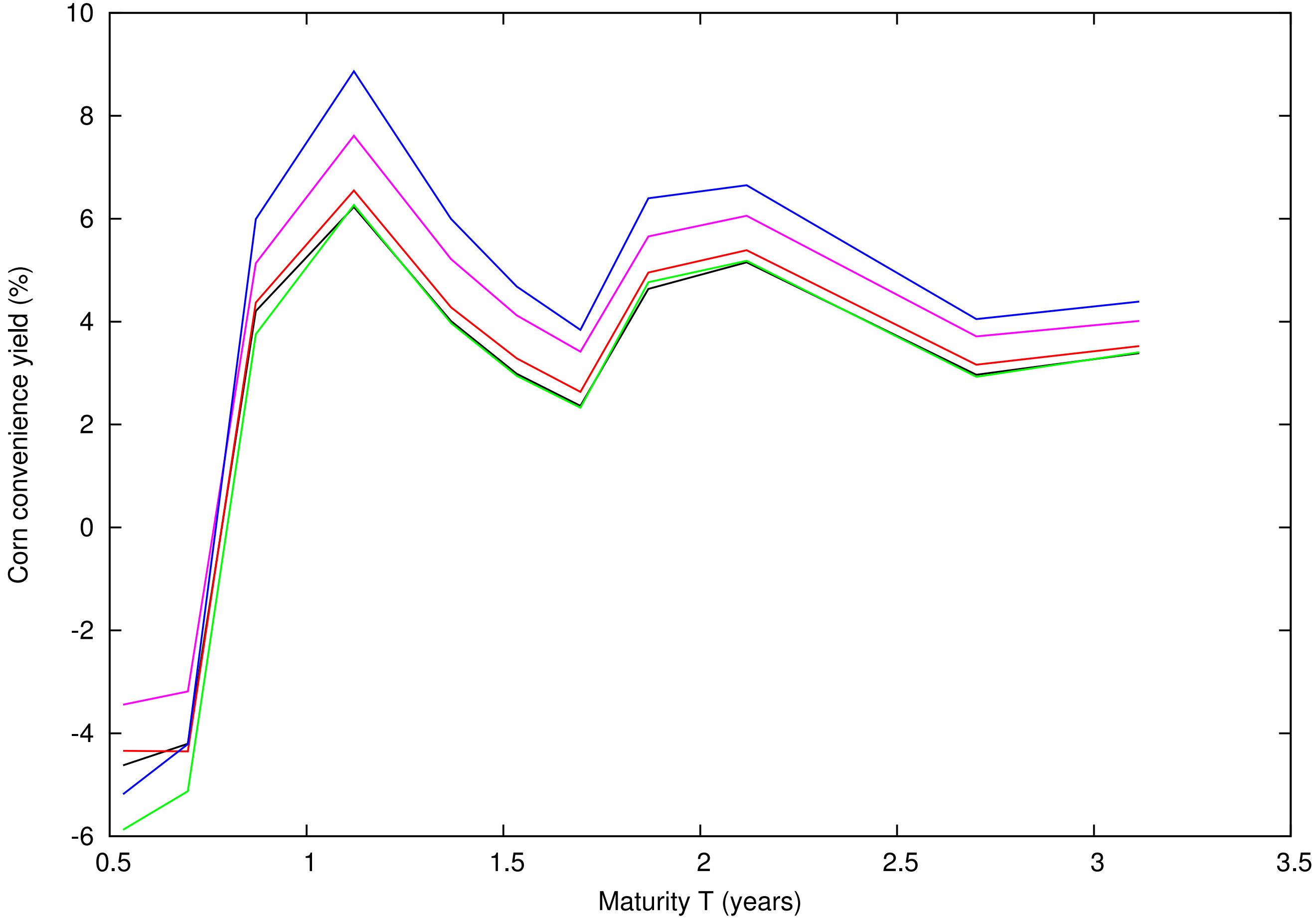
Few convenience-yield curves for Copper



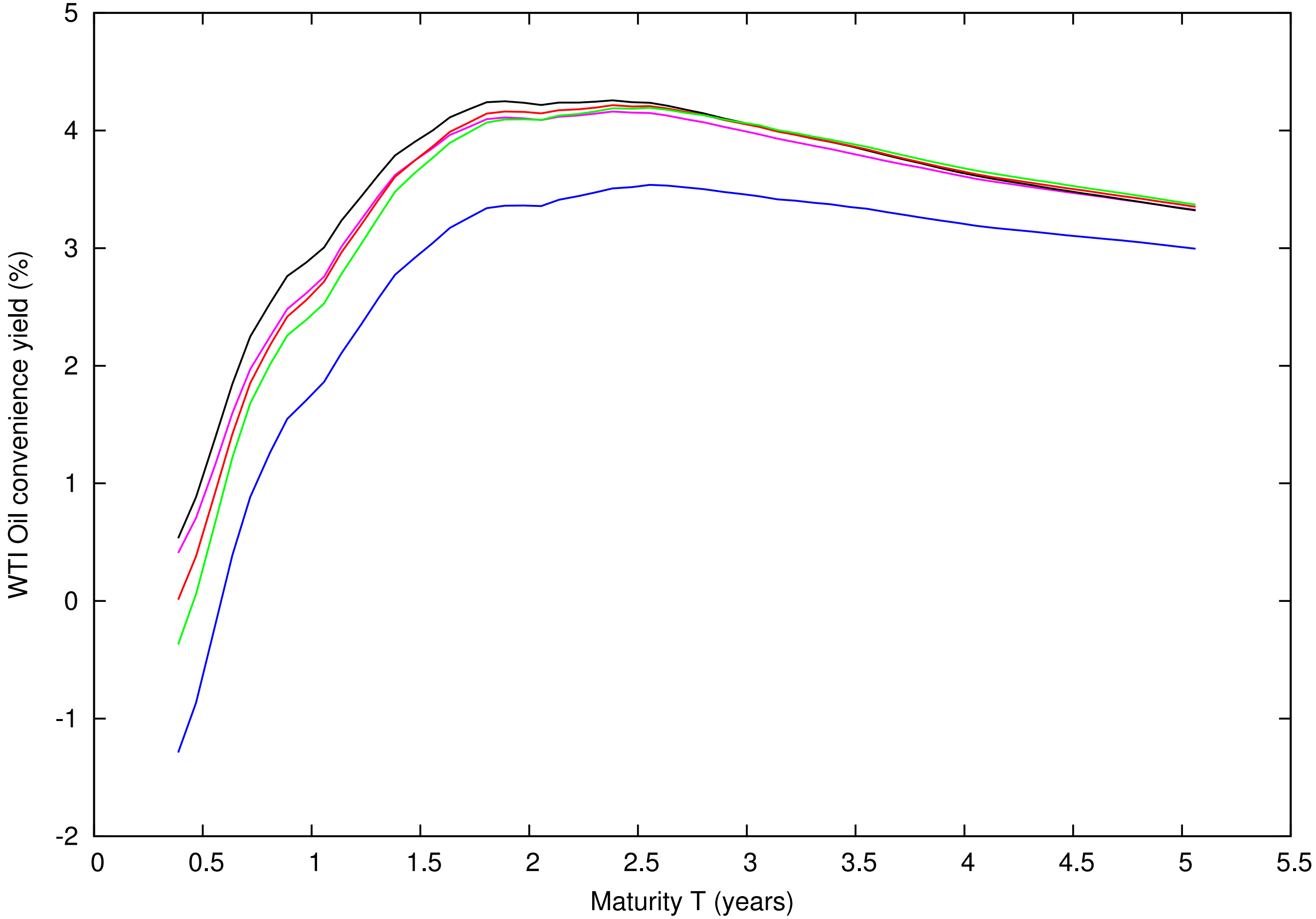
Recall the zero interest-rate curves



Few convenience-yield curves for Corn



Few convenience-yield curves for WTI Oil



NPV of a commodity forward contract

Given the convenience yield curve $y(T)$ define the (continuously-compounded) **convenience discount factor** $D^c(T)$ as

$$D^c(T) = e^{-y(T)\,T} \tag{28}$$

so that the **commodity forward price** can be written as

$$f(T) = S\frac{D^c(T)}{D^r(T)}\,. \tag{29}$$

The commodity forward contract NPV is then given by

$$\text{NPV} = W \cdot D^s(T) \cdot \Big[f(T) - K\Big] \tag{30}$$

At inception NPV=0 implies $K = f(T)$.

Commodities as currencies

Recall that given two currencies € and \$ and their exchange rate $X_{€\$}$

$$1\,\$ = \frac{1€}{X_{€\$}}$$

we have the arbitrage-free forward exchange rate

$$X_{€\$}(T) = X_{€\$}\,\frac{D^{\$}(T)}{D^{€}(T)}$$

To be compared with the commodity relationship

$$f(T) = S\,\frac{D^{c}(T)}{D^{r}(T)}$$

Questions?

Generic pricing formula (1/2)

Remember the discount factor relationship with the money account?

$$D(T) = E\left[e^{-\int_0^T r(t)\mathrm{d}t}\right] = E\left[e^{-r_1\,\tau_1 - \ldots - r_n\,\tau_n}\right] \tag{31}$$

If one payment C is made at a future date T we have,

$$PV = D(T)\,C = E\left[e^{-\int_0^T r(t)\mathrm{d}t}\,C\right] \tag{32}$$

Assume now n deterministic cash flows $C_1, C_2, \ldots, C_n$, one each day, i.e. at dates $T_1, T_2, \ldots, T_n$

$$PV = D(T_1)\,C_1 + D(T_2)\,C_2 + D(T_3)\,C_3 + \ldots \tag{33}$$

Generic pricing formula (2/2)

For example, in the simple case with only three dates we have

$$\begin{aligned}
PV &= E\left[e^{-r_1\tau_1} C_1 + e^{-r_1\tau_1 - r_2\tau_2} C_2 + e^{-r_1\tau_1 - r_2\tau_2 - r_3\tau_3} C_3\right] \\
&= e^{-r_1\tau_1} \left\{C_1 + E\left[e^{-r_2\tau_2}\left(C_2 + e^{-r_3\tau_3} C_3\right)\right]\right\} \\
&= e^{-r_1\tau_1} \left\{C_1 + \sum_{r_2} P(r_2)\left[e^{-r_2\tau_2} E\left(C_2 + e^{-r_3\tau_3} C_3 | r_2\right)\right]\right\} \\
&= e^{-r_1\tau_1} \left\{C_1 + \sum_{r_2} P(r_2)\left[e^{-r_2\tau_2}\left(C_2 + \sum_{r_3} P(r_3|r_2) e^{-r_3\tau_3} C_3\right)\right]\right\}
\end{aligned}$$

Wouldn't it be great if we could do the same for random cash flows?

Pricing in absence of arbitrage

There is a fundamental result from Harrison and Kreps (1979) that holds for stochastic cash flows

Theorem: in absence of arbitrage, the generic pricing formula for an option with a payoff $H(T, r_t)$, depending on the interest rates r_t, is

$$PV = E\left[e^{-\int_0^T r_t \mathrm{d}t}\, H(T, r_t)\right] \tag{34}$$

where E is the *expected-value operator* on the risk-neutral measure.

In terms of the money-market account $M(t)$, recall $M(0)$, we have

$$PV = M(0)\, E\left[\frac{H(T, r_t)}{M(T)}\right] \tag{35}$$

Numeraires

It turns out that formula (35) is a special case of a general theorem that holds for generic numeraire assets.

A *numeraire* is any tradable asset that

- is always positive
- does not pay any dividends (nor coupons)

Examples: risk-less money-market account, risk-less zero-coupon bond, the price of a commodity (e.g. the gold price). See Hull book for more examples.

Equivalent Martingale Measure

Theorem:
A continuous economy is arbitrage-free and every security is attainable if for every choice of numeraire there exists a unique equivalent (martingale) measure. The security PV is then given by

$$PV = N(0)\, E^N \left[\frac{H(T, r_t)}{N(T)} \right] \tag{36}$$

- A security is attainable if it can be replicated (recall stock options replication with delta-hedging)
- The expectation E^N is taken according to the equivalent measure

Change of Numeraire

How do we compute the expectation in (36)? E.g. using the following theorem.

Given two numeraires M and N and a financial quantity G,

$$E^M[G] = E^N\left[G\frac{M(T)\,N(0)}{M(0)\,N(T)}\right] \tag{37}$$

- This result was first proved by Geman et al. in 1995
- It is incredibly useful in practice

Numeraire applications: payoff at maturity (1/2)

Consider the numeraire associated to a payoff of 1$ at time T ($N(0) = D(T)$, $N(T) = 1$) and E^T the corresponding expectation, defining G as

$$G(T) = \frac{M(0)}{M(T)} H(T) \tag{38}$$

the forward payoff corresponding to H, then

$$PV = E^M [G(T)] = E^T \left[G(T) \frac{M(T)\, N(0)}{M(0)\, N(T)} \right] \tag{39}$$

so that ...

Numeraire applications: payoff at maturity (2/2)

.. we obtain

$$PV = D(T)\, E^{T}\left[H(T)\right] \tag{40}$$

Notice how the stochastic rates disappeared.

Other examples (see Hull's book)

- Choosing N as the period-compounded deposit yields the correct measure for caplets and floorlets
- Choosing the fixed-rate annuity A_{fixed} as numeraire provides the measure used to price swaptions

Questions?

Recall the Brownian motion

A continuous stochastic process W_t satisfying,

- W_0=0
- $W_t - W_s$ is independent form W_s (for $0 < s < t$)
- $W_t - W_s$ is normally distributed, precisely as $N(0, t - s)$

is a standard Brownian motion.

Complex stochastic processes can be built upon the Brownian motion using stochastic differential equations.

Stochastic differential equations (1/3)

A *stochastic differential equation* for r_t is defined as

$$\mathrm{d}r_t = u(r_t, t)\mathrm{d}t + \sigma(r_t, t)\mathrm{d}W_t \tag{41}$$

- u is the generic drift (deterministic)
- σ is the generic volatility (deterministic)
- W_t is a Brownian motion (stochastic)

Every stochastic differential equation is really a shorthand for the following stochastic (Ito's) integral equation

$$r_t - r_0 = \int_0^t u(r_s, s)\mathrm{d}s + \int_0^{W_t} \sigma(r_s, s)\mathrm{d}W_s \tag{42}$$

Stochastic differential equations (2/3)

We can integrate the stochastic integral equation over a short time interval $[t_1, t_2]$. The drift term becomes

$$u(r_1, t_1)(t_2 - t_1) \tag{43}$$

Since W_t is a Brownian motion the volatility term becomes

$$\sigma(r_1, t_1)\,(W_1 - W_2) = \sigma(r_1, t_1)\sqrt{t_2 - t_1}\,\varepsilon \tag{44}$$

where ε is a *Gaussian random* number

$$\varepsilon \sim N(0, 1) \tag{45}$$

Stochastic differential equations (3/3)

In summary the stochastic differential equation for r_t

$$\mathrm{d}r_t = u(r_t, t)\mathrm{d}t + \sigma(r_t, t)\mathrm{d}W_t \tag{46}$$

can be interpreted, for a small interval $[t, t + \Delta t]$, as a *simulation* for the future values of $r_{t+\Delta t}$ given the value r_t

$$r_{t+\Delta t} - r_t = u(r_t, t)\Delta t + \sigma(r_t, t)\ \sqrt{\Delta t}\ \varepsilon_j \tag{47}$$

with many samples ε_j's taken "Normally" randomly

Example: Martingales

In the particular case of processes where $u = 0$,

$$\mathrm{d}X_t = \sigma(X_t, t)\mathrm{d}W_t \tag{48}$$

we have a *Martingale*.

Taking an expectation of any martingale yields the current value

$$E\left[X_t\right] = X_0 \tag{49}$$

Choosing $X_t = Y_t/N_t$ leads to the numeraire pricing equation

$$Y_0 = N_0\, E\left[\frac{Y_t}{N_t}\right] \tag{50}$$

Example: Hull–White SDE

Consider a Stochastic differential equation describing a financial model. For example the Hull-White short-rate process defined by

$$\mathrm{d}r_t = [\theta(t) - a\, r_t]\ \mathrm{d}t + \sigma\, \mathrm{d}W_t \tag{51}$$

Consider an option V so that the underlying short rate satisfies the Hull-White model. Proceeding with no-arbitrage arguments and using Ito's lemma it can be shown that V satisfies

$$\frac{\partial V}{\partial t} + [\theta(t) - a\, r]\, \frac{\partial V}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial r^2} - r\, V = 0 \tag{52}$$

which is the analogous of the Black-Scholes equation for the Hull-White model

Partial differential equations

For every model described by a stochastic differential equation, the corresponding option price V satisfies a partial differential equation

$$\frac{\partial V}{\partial t} - r\,V + u(r,t)\frac{\partial V}{\partial r} + \frac{\sigma(r,t)^2}{2}\frac{\partial^2 V}{\partial r^2} = 0 \tag{53}$$

- r is the source term
- u is the drift (a velocity field)
- σ is the volatility (the diffusion coefficients)

Equation (53) is a parabolic partial differential equation, with the appropriate boundary and final conditions can be solved to given $V(r,T)$

Feynman-Kac formula

Richard Feynman (physicist) and Mark Kac (mathematician) demonstrated a result that applied to the equation

$$\frac{\partial V}{\partial t} - r\,V + u(r,t)\frac{\partial V}{\partial r} + \frac{\sigma(r,t)^2}{2}\frac{\partial^2 V}{\partial r^2} = 0 \tag{54}$$

with the final condition, defined by the payoff H

$$V(T,r) = H(r) \tag{55}$$

Has a solution

$$V(t,r) = E\left[e^{-\int_t^T r(s)\mathrm{d}s}\,H(r)\right] \tag{56}$$

The Feynman-Kac formula is another derivation of the generic price formula. Most numerical methods use this equation.

Questions?

Numerical methods for interest-rate models

Even the most sophisticated interest-rate model is useless if it cannot produce at least a single number

Definition 1:
An *interest-rate model* is a mathematical tool that describes interest rates in the financial markets.

Definition 2:
A *numerical method* is an algorithm that is applied to a model in order to compute numerical values for the financial variables

Analytical formulas

Analytical formulas are historically the first numerical method used to compute actual numbers from financial models

In few rare and exceptional cases it is possible to find analytical solutions to derivatives models. Important examples are

- Merton formula for Black-Scholes equations
- Formulas for barrier and digital options priced in Black-Scholes model
- Discount-factor and bond-option formulas for Hull-White model

Analytical approximations (1/2)

Even the most basic analytical solutions needs numerical methods to be evaluated. For example the cumulative normal distribution

$$N(x) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t \tag{57}$$

needs to be approximated. A simple approximation is given by

$$N(x) \sim 1 - \frac{1}{\sqrt{2\,\pi}} e^{-x/2} \left(a_1\,k + a_2\,k^2 + a_3\,k^3 \right) \tag{58}$$

with

$$k = \frac{1}{1 + 0.33267\,x} \tag{59}$$

and $a_1 = 0.4361836$, $a_2 = -0.1201676$, and $a_3 = 0.937298$

Analytical approximations (2/2)

Most analytical formulas are obtained from the mathematical theory of *analytical functions*, i.e. functions that can be obtained from a locally-convergent *power series*

A fairly exhaustive collection of analytical formulas for option pricing has been compiled by *the collector** and can be found in

- *The Complete Guide to Option Pricing Formulas*, Espen Gaarder Haug, Mc Graw Hill (from first edition)

*See also the interesting *picture* collection

Questions?

References

- *Options, future, & other derivatives*, John C. Hull, Prentice Hall (from fourth edition)

- *Efficient methods for valuing interest rate derivatives*, Antoon Pelsser, Springer Finance

- *Interest rate models: theory and practice*, D. Brigo and F. Mercurio, Springer Finance (from first edition)

- The collector web site: `http://www.espenhaug.com`