Equilibrium interest-rate models

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November 17th, 2012

Lecture Summary

- Bond options
- Interest-rate models
- Short-rate models
- Equilibrium models: Vasicek model
- Other equilibrium models
- Binomial and trinomial trees
- Finite differences

Bond options

A bond option is an option that at a certain maturity time T^o gives the right to purchase a bond at a pre-determined price P_K. Thus the payoff of a bond option is given by

$$H^{\pm}(P, P_K) = N\,Q\,\max\left[0, \pm(P - P_K)\right] \tag{1}$$

H^+ is used for the call option and H^- for the put option. Q is called the parity.

- The bond maturity is usually much later than the option maturity
- Bond options are often quoted as yield volatility
- Quoted bond options are commonly on fixed-rate government bonds

Zero coupon bond options (1/3)

Consider a zero-coupon-bond option maturing at T_1 on a risk-free* zero-coupon bond with a maturity at T_2 (set $T = T_2 - T_1$).

Since the forward bond price can be estimated to be

$$P = P(T_1, T_2) = D(T_2)/D(T_1) \tag{2}$$

we have the following

Theorem:

Any call option on a zero-coupon bond can be transformed into a floorlet and any put option can be transformed into a caplet. The caplet/floorlet tenor is given by the bond residual maturity.

*Models only: risk-less zero-coupon bonds are not existent on the market

Zero coupon bond options (2/3)

Proof:

For call options the terms inside the pricing formulas

$$
\begin{aligned}
D(T_1)\, H^{\pm} &= N\, Q\, D(T_1) \max\left[0, \frac{D(T_2)}{D(T_1)} - P_K\right] \\
&= N\, Q \max\left[0, D(T_2) - D(T_1)\, P_K\right] \\
&= N\, Q\, P_K D(T_2) \max\left[0, \frac{1}{P_K} - 1 + 1 - \frac{D(T_1)}{D(T_2)}\right] \\
&= N\, D(T_2) \max\left[0, \frac{1}{T}\left(\frac{1}{P_K} - 1\right) - \frac{1}{T}\left(\frac{D(T_1)}{D(T_2)} - 1\right)\right] \\
&= N\, D(T_2) \max\left[0, r_k - r_{fwd}(T_1, T_2)\right]
\end{aligned}
$$

Zero coupon bond options (3/3)

where we set

$$r_K = \frac{1}{T}\left(\frac{1}{P_K} - 1\right) \tag{3}$$

and

$$Q = \frac{1}{T\,P_K} \tag{4}$$

similarly for put options

Using the generic pricing formula and the change of numeraire we obtain the result.

Models with known bond options

Interest-rate models for which it is known an analytic, or simplified form, for pricing bond options are extremely useful

- Because the equivalence of bond options and Caplets/Floorlets we can easily compute the prices for Caps and Floors, therefore simplifying the calibration procedures

- For a number of models (unfortunately not that many) there exist a technique, namely the Jamshidian decomposition, that expresses the swaptions prices in terms of a sum of options on zero-coupon bonds

Questions?

Interest rate models

An interest-rate model is a mathematical tool that describes interest rates in the financial markets.

Any interest-rate model should provide

- the shape of the discount factor
- calibration tools to match market variables
- a method, numerical or analytical, to price interest-rate derivatives

Example:
Discount-rate bootstrap, cash-flow discounting, and Black formula

Stochastic interest rate models

Stochastic interest-rate model use random variables to describe the unknown factors determining interest rates

- Single factor models: when only one random factor describes the market uncertainties (usually easy to implement, however may not be describe well all instruments)

- Multi-factor models: two, three, four, ..., random factors used to describe markets (hard to implement and calibrate, great results)

Note: there might be problems when too many factors are used (over calibration, missing arbitrage opportunities)

Spot and Forward rate models

- **Spot-rate models** consider the dynamic of the short-rate to determine the dynamics of the whole interest-rate curve. E.g., Vasicek, Hull-White, Black-Karasinski

- **Forward-rate models** directly model the forward rates. E.g., Black model for Caps, (forward) Libor-market model, swap-market model

Spot rate models

The dynamic of the whole curve is given by changes in the spot rate

$$
\begin{aligned}
PV &= E\left[e^{-r_1\tau_1}C_1 + e^{-r_1\tau_1 - r_2\tau_2}C_2 + e^{-r_1\tau_1 - r_2\tau_2 - r_3\tau_3}C_3 + \ldots\right] \\
&= e^{-r_1\tau_1}\left\{C_1 + E\left[e^{-r_2\tau_2}\left(C_2 + e^{-r_3\tau_3}C_3 + \ldots\right)\right]\right\} \\
&= e^{-r_1\tau_1}\left\{C_1 + \sum_{r_2} P(r_2)\left[e^{-r_2\tau_2}E\left(C_2 + e^{-r_3\tau_3}C_3|r_2\right) + \ldots\right]\right\} \\
&= e^{-r_1\tau_1}\left\{C_1 + \sum_{r_2} P(r_2)\left[e^{-r_2\tau_2}\left(C_2 + \sum_{r_3} P(r_3|r_2)e^{-r_3\tau_3}C_3\right) + \ldots\right]\right\}
\end{aligned}
$$

Taking the limit for $\tau_i \to 0$ we obtain a the PV of any asset.

Equilibrium models

Drift and volatility depend only on interest rates (not time)

$$\mathrm{d}r_t = \mu(r_t)\,\mathrm{d}t + \sigma(r_t)\,\mathrm{d}W_t \tag{5}$$

Example: simple model with log-normal short-term interest rates with μ and σ proportional to r_t:

$$\mathrm{d}r_t = a\,r_t\,\mathrm{d}t + \sigma\,r_t\,\mathrm{d}W_t \tag{6}$$

proposed by Rendleman and Bartter in 1980.

- Rates are always positive
- Rates are **not** mean reverting

Mean reverting models

Even though interest rates may increase or decrease ...

- when are too high for too long it is hard to borrow money, hence business tends not to be financed well, creating a recession. E.g.: beginning of the ’80s.
- when are too low for too long every business receives the money that is needed, over-investment is common, the economy heats up and interest rates rise because inflationary pressure. E.g.: 2002-2006

Models that intrinsically capture these features are called mean reverting models.

Questions?

Vasicek model

The simplest mean reverting short-term interest rates model satisfying

$$\mathrm{d}r_t = \mu(r_t, t)\mathrm{d}t + \sigma(r_t, t)\mathrm{d}W_t \tag{7}$$

was proposed by Vasicek in 1977,

$$\mathrm{d}r_t = a(b - r_t)\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{8}$$

- Rates are mean reverting to level b
- Rate increments are “normally” distributed
- Note $r_t\,\mathrm{d}t = b\,\mathrm{d}t - a^{-1}\,\mathrm{d}r_t + (\sigma/a)\,\mathrm{d}W_t$

Vasicek discount curve

The discount factor can be computed analytically,

$$D(T) = E\left[e^{-\int_0^T r_t dt}\right] = A(T)e^{-B(T)r_0} \tag{9}$$

where

$$B(T) = \frac{1 - e^{-aT}}{a} \tag{10}$$

and

$$A(T) = \exp\left\{\frac{[B(T) - T](a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(T)^2}{4a}\right\} \tag{11}$$

Only three (four with r_0) parameters fits all discount curves

Vasicek bond options (1/2)

In the Vasicek model it is also possible to compute analytically the value of a European option on a zero-coupon bond. Given the payoff

$$H^{\pm}(P) = \max\left[0, \pm\left(P(T,S) - X\right)\right] \tag{12}$$

where X is the strike, T the option maturity on a bond with a maturity at S. Let

$$\sigma_p = \frac{\sigma}{a}\left[1 - e^{-a(S-T)}\right]\sqrt{\frac{1-e^{-2\,a\,T}}{2\,a}} \tag{13}$$

and

$$h = \frac{\sigma_p}{2} + \frac{1}{\sigma_p}\ln\frac{D(S)}{X\,D(T)} \tag{14}$$

Vasicek bond options (2/2)

it was shown by Jamshidian that the option premium is given by

$$BO^{\pm} = \pm D(S)\, N(\pm h) - (\pm) X\, D(T)\, N(\pm(h - \sigma_p)) \tag{15}$$

This expression and the equivalence of bond options and optionlets (caplets and floorlets) allows us to calibrate Vasicek model analytically on volatilities of caps and floors

This expression and the Jamshidian decomposition of swaptions allows us to calibrate Vasicek model analytically on swaption volatilities

Vasicek: the good, ...

We like it because

- mean-reverting model
- simple, easy to understand
- analytical formulas for discount factor and bond options

Vasicek: the bad

However, we would like

- to fit the yield curve with more than 3 parameters
- to fit volatility and yield curves independently
- more than one factor to price complex instruments
- interest rates not to go negative (with such a high probability)

Questions?

C-I-R model

In order to avoid the problem of negative interest rates, Cox, Ingersoll, and Ross proposed, in 1985,

$$\mathrm{d}r_t = a(b - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t \tag{16}$$

On those paths where interest rates are close to zero, the stochastic contribution goes to zero.

- Rates are mean reverting to level b
- Discount factor and bond options can be easily computed
- Jamshidian decomposition is working
- Same problems as Vasicek's model (other than non-negative rates)

Shortcomings of equilibrium models

- In the equilibrium models the discount factor was an output of the model.
- Discount factors are not consistent with market
- A small difference in the forward curve results in a big difference in the Cap/Floor value (same for swaptions)

We need a whole new category of models that can exactly fit the current discount curve

Questions?

Numerical methods for short-rate models

Vasicek SDE

The stochastic differential equation assumed in the Vasicek model is

$$\mathrm{d}r_t = [b - a\,r_t]\;\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{17}$$

Consider an option V to be priced according to this model. Proceeding with no-arbitrage arguments and using Ito's lemma it can be shown that V satisfies

$$\frac{\partial V}{\partial t} + [b - a\,r]\,\frac{\partial V}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial r^2} - r\,V = 0 \tag{18}$$

which is the analogous of the Black-Scholes equation for the Vasicek model

Solving partial differential equations

For every model described by a stochastic differential equation, the corresponding option price V satisfies a partial differential equation

$$\frac{\partial V}{\partial t} + s(r,t)\,V + u(r,t)\frac{\partial V}{\partial r} + \frac{\sigma(r,t)^2}{2}\frac{\partial^2 V}{\partial r^2} = 0 \qquad (19)$$

Note that s is the source term; u is the drift (a velocity field); σ is the volatility (the diffusion coefficients).

Existence and uniqueness of solution

Theorem Under reasonable assumptions on the coefficients s, u, and σ the solution of equation (19) (a parabolic partial differential equation) is guaranteed given appropriate boundary and final conditions.

For the proof one can, for example, use the Feynman-Kac formula.

We are going to use different finite-difference approaches to solve equation (19): binomial trees, trinomial trees, and proper finite-difference methods.

Binomial trees

Given a stochastic equation, e.g.,

$$\mathrm{d}r_t = [\theta - a\,r_t]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{20}$$

at each time step, from one state only two states can be reached

$$r(t_{i+1}) = \begin{cases} r(t_i) + \Delta_u & \text{with probability } p_u \\ r(t_i) + \Delta_d & \text{with probability } p_d \end{cases}$$

determine Δ_u, Δ_d, p_u, and p_d to be consistent with the dynamics

Starting from $r(T_0)$, build a binomial tree for the short-term interest rate until reaching the maturity of the longest instrument

Example: binomial tree for bond options (1/2)

Computation of bond option maturing at T_1 on a zero-coupon bond with a maturity T_2.

Step 1: The bond price $B_k(T_2) = 1$ is known at all nodes

Step 2: Roll back the bond price from T_2 to T_1, at each node using the local *instantaneous forward rate*

$$B_k(t_i) = e^{-r_k(t_i)\,\Delta t}\left[p_u\,B^u_{k+1}(t_{i+1}) + p_d\,B^d_k(t_{i+1})\right] \tag{21}$$

Now we have a value for the zero-coupon bond $B_k(T_1)$ at each node

Example: binomial tree for bond options (2/2)

Given the option strike K,

Step 3: Compute the option price P_k at T_1 at each node

$$P_k(T_1) = \max[0, B_k(T_i) - K] \tag{22}$$

Step 4: Roll back the option price at from T_1 to T_0, properly discounting at each node, to obtain the current option price $P_0(T_0)$

Binomial trees

+ $N_{\text{nodes}} = N_{\text{steps}} + 1$
+ Easy to implement
− Slow convergence, often unstable needs small time steps

Trinomial trees

Trinomial trees assume, at each time step, that three states can be reached

$$r(T_{i+1}) = \begin{cases} r(T_i) + \Delta_u & \text{with probability } p_u \\ r(T_i) + \Delta_m & \text{with probability } p_m \\ r(T_i) + \Delta_d & \text{with probability } p_d \end{cases}$$

Discount at each node is performed using three terms:

$$B_k(t_i) = e^{-r_k(t_i)\,\Delta t}\left[p_u\, B^u_{k+2}(t_{i+1}) + p_m\, B^m_{k+1}(t_{i+1}) + p_d\, B^d_k(t_{i+1})\right]$$

- $N_{\text{nodes}} = 2\,N_{\text{steps}} + 1$
- Still not that hard to implement
- Better convergence than binomial trees
- Stability rules are well known (Courant condition)

Questions?

Finite differences

The *diffusion* equation (19)

$$\frac{\partial V}{\partial t} = \left[-s - u \frac{\partial}{\partial r} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial r^2} \right] V \tag{23}$$

is written in terms of the discretized operator $\mathcal{A}$:

$$\frac{\partial V}{\partial t} = \mathcal{A} V \tag{24}$$

- Use *spatial* discretization to approximate the operator $\mathcal{A}$
- Use a time-evolution scheme to for time dependence

Finite differences: *spatial* discretization (1/2)

Given a fixed time t, consider the discretized asset price $V_k = V(r_k, t)$ on a grid of equally spaced rates

$$r_k = k\,\Delta \qquad \text{for} \quad k = 1, \ldots, N_{\text{nodes}} \tag{25}$$

Using the second-order Taylor approximation we have

$$V_{k+1} \;=\; V_k + \frac{\partial V}{\partial r}\Delta + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\Delta^2 + \varepsilon(\Delta^3) \tag{26}$$

$$V_{k-1} \;=\; V_k - \frac{\partial V}{\partial r}\Delta + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\Delta^2 + \varepsilon(\Delta^3) \tag{27}$$

Finite differences: *spatial* discretization (2/2)

Neglecting terms of order $\varepsilon(\Delta^3)$, first sum and the subtract equations (26) and (27) to obtain

$$\frac{\partial V}{\partial r} \simeq \frac{\delta_k V}{\Delta} = \frac{V_{k+1} - V_{k-1}}{2\,\Delta} \tag{28}$$

$$\frac{\partial^2 V}{\partial r^2} \simeq \frac{\delta_k^2 V}{\Delta^2} = \frac{V_{k+1} - 2\,V_k + V_{k-1}}{\Delta^2} \tag{29}$$

so that the discretized version of operator $\mathcal{A}$ becomes

$$\mathcal{A}_k\, V = -sV_k - \frac{u}{\Delta}\,(\delta_k V) - \frac{\sigma^2}{2\,\Delta^2}\left(\delta_k^2 V\right) \tag{30}$$

Finite differences: time evolution (1/3)

Time evolution on a single step can be obtained discretizing the time-derivative operator going backward from time t_{i+1} to t_i

Explicit method (operator $\mathcal{A}$ applied to previous step)

$$\frac{V^{i+1} - V^i}{\Delta t} = \mathcal{A}\, V^{i+1} \qquad \Longleftrightarrow \qquad V^i = V^{i+1} - \Delta t\ \mathcal{A}\, V^{i+1}$$

- Equivalent to trinomial tree
- Conditionally stable (Courant condition for stability)
- Easy to implement as there are no matrix inversions
- First order in time

Finite differences: time evolution (2/3)

Full implicit method

operator $\mathcal{A}$ applied to current step

$$\frac{V^{i+1} - V^i}{\Delta t} = \mathcal{A} V^i \quad \Longleftrightarrow \quad (I + \Delta t\, \mathcal{A}) V^i = V^{i+1}$$

We need to solve an equation to obtain V^i from V^{i+1}

- Much better than trinomial tree (faster convergence)
- Unconditionally stable
- Not so easy to implement as **there is** matrix inversions
- First order in time

Finite differences: time evolution (3/3)

Crank-Nicolson method

Space operator is equally split between the two time steps

$$\frac{V^{i+1} - V^i}{\Delta t} = \frac{1}{2}\left[\mathcal{A}V^i + \mathcal{A}V^{i+1}\right]$$

We still need to solve an equation to obtain V^i from V^{i+1}

- As hard to implement as the full implicit method
- Unconditionally stable
- Not so easy to implement as **there is** matrix inversions
- Second order in time

Finite differences: everything together

In the finite-difference framework the *diffusion* equation

$$\frac{\partial V}{\partial t} = -s\,V - u\frac{\partial V}{\partial r} - \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial r^2} \tag{31}$$

can be discretized, for example using the explicit method, as

$$\frac{V^{i+1} - V^{i}}{\Delta t} = -s\,V_k^{i+1} - \frac{u}{\Delta}\left(\delta_k V^{i+1}\right) - \frac{\sigma^2}{2\,\Delta^2}\left(\delta_k^2 V^{i+1}\right) \tag{32}$$

Starting from maturity one can rollback all assets, apply the conditions at the important nodes, and compute the asset values at T_0

Questions?

References

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- *Interest rate models: theory and practice*, D. Brigo and F. Mercurio, Springer Finance (from first edition)

