Market interest-rate models

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Lecture Summary

- No-arbitrage models
- Detailed example: Hull-White
- Monte Carlo simulations
- Libor market model
- Other advanced models

Equilibrium models

Drift and volatility depend only on interest rates (not time)

$$\mathrm{d}r_t = \mu(r_t)\,\mathrm{d}t + \sigma(r_t)\,\mathrm{d}W_t \tag{1}$$

+ Great for macroeconomic studies
- Discount factor and volatility curves are in parametric form
- There is arbitrage! (Cannot be used in market transactions.)

No arbitrage models

In these models, also known as term-structure-fitting models, there can not be arbitrage because of the "wrong" discount factor.

Simplest example:

$$\mathrm{d}r_t = \theta(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{2}$$

where the function $\theta(t)$ can be determined so that the discount factor is modeled exactly.

This model was proposed by Ho and Lee in 1986 in the form of a binomial tree. Problem: no mean reversion!

Spot and Forward rate models

There can be two types of no-arbitrage models

- **Spot-rate models** consider the dynamic of the short-rate to determine the dynamics of the whole interest-rate curve. E.g., Vasicek, Hull-White, Black-Karasinski

- **Forward-rate models** directly model the forward rates. E.g., Black model for Caps, (forward) Libor-market model, swap-market model

Hull-White model

In the original formulation Hull & White, in 1991, proposed a time-dependent evolution for short rates,

$$\mathrm{d}r_t = [\theta(t) - a(t)\, r_t]\ \mathrm{d}t + \sigma(t)\, \mathrm{d}W_t \tag{3}$$

where $\theta(t)$, $a(t)$, and $\sigma(t)$ are functions of time. This formulation has too many free parameters and, while perfectly calibrating discount factor and volatilities, is not analytically tractable.

Hence Hull and White proposed the simplification

$$\mathrm{d}r_t = [\theta(t) - a\, r_t]\ \mathrm{d}t + \sigma\, \mathrm{d}W_t \tag{4}$$

where $\theta(t)$ is a time function and a, and σ are constants

Hull-White solution (1/3)

Consider a stochastic process x_t and a deterministic function $y(t)$, let

$$r_t = x_t + y(t) \qquad \Rightarrow \qquad \mathrm{d}r_t = \mathrm{d}x_t + \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t \tag{5}$$

where we used Ito's lemma. Substituting in (4),

$$\mathrm{d}x_t + \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t = \theta(t)\mathrm{d}t - a\,x_t\,\mathrm{d}t - a\,y(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{6}$$

Assume $y(t)$ to satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \theta(t) - a\,y \tag{7}$$

Hull-White solution (2/3)

We obtain the stochastic differential equation for x_t

$$\mathrm{d}x_t = -a\,x_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{8}$$

which can be integrated to give

$$x_t = x_0\,e^{-a\,t} + \sigma\int_0^t e^{-a(t-s)}\mathrm{d}W_s \tag{9}$$

Computing the discount factor and solving for $y(t)$ we obtain

$$y(t) = f(t) + \frac{\sigma^2}{2\,a^2}\left(1 - e^{-a\,t}\right) \tag{10}$$

where f is the market instantaneous forward rate at time t

Hull-White solution (3/3)

Bringing all the pieces together we have

$$\begin{aligned} r_t &= x_t + y(t) = \\ &= f(t) + [r_0 - f(0)]\, e^{-a\,t} + \frac{\sigma^2}{2\,a^2}\left(1 - e^{-a\,t}\right) + \sigma \int_0^t e^{-a(t-s)} \mathrm{d}W_s \end{aligned}$$

We can compute average and variance

$$\langle r_t \rangle = r_0 + f(t) - f(0) + \frac{\sigma^2}{2\,a^2}\left(1 - e^{-a\,t}\right) \tag{11}$$

$$\left\langle (r_t - \langle r_t \rangle)^2 \right\rangle = \frac{\sigma^2}{2\,a}\left(1 - e^{-2a\,t}\right) \tag{12}$$

Features of Hull-White solution

- The *homogeneous solution* x_t follows a Vasicek model with θ=0 and the same a, σ

- For small t rates are centered around r_0 and follow a Brownian motion (variance increase like t)

- $\tau = 1/a$ is the time scale of volatility increase

- Interest rates are *almost* normally distributed around the instantaneous forward rate

Nice features of Hull-White model

Like the Vasicek model, the Hull-White model is analytically tractable

- Mean-reverting model
- Known price for a zero-coupon bond
- Discount curve obtained exactly
- Known price for options on a zero-coupon bond
- Swaptions are computable using Jamshidian decomposition
- Easy to calibrate on quoted volatilities for Caps/Floors or Swaptions (need to choose one or the other)
- Quite easy to be solved numerically for American-style options (e.g., using a trinomial tree)

Not so nice features of Hull-White model

The Hull-White model has some drawbacks

- Only two free parameters to calibrate all the volatilities
- The *hump* in the caplet volatilities is **not** observed
- Only one factor: not suitable for some exotic options
- Rates can become negative with a positive probability

Black-Karasinski model

To avoid negative interest rates, we model the short log rates:

$$x_t = \log r_t \qquad \text{if and only if} \qquad r_t = e^{x_t} \tag{13}$$

$$\mathrm{d}x_t = [\theta(t) - a\, x_t]\, \mathrm{d}t + \sigma\, \mathrm{d}W_t \tag{14}$$

Proposed by Black and Karasinski in 1991 as a generalization of the Black-Derman-Toy model.

\+ Interest rates are positive and mean reverting
– Discount rates **do not** have an analytical form
– Bond options cannot be evaluated analytically
– Price of Caps/Floors/Swaptions are poorly interpolated

Questions?

Monte Carlo integrals (1/2)

Monte Carlo methods are based on the analogy between *measure theory*, used in the Lebesgue formulation of integrals, and *probability theory*: an average can be computed as an integral

Given a random variable x, uniformly distributed on $[0, 1]$, and a function f

$$E[f(x)] = \int_0^1 f(x)\,\mathrm{d}x \tag{15}$$

consider N samples for x, namely $x_1, \ldots, x_N$, then

$$E[f(x)] = \frac{1}{N} \sum_{k=1}^{N} f(x_k) \tag{16}$$

Monte Carlo integrals (2/2)

Expressions (15) and (16) together give

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{N}\sum_{k=1}^{N} f(x_k) \tag{17}$$

This is the fundamental result used in most Monte Carlo methods.

Monte Carlo simulations compute integrals

Reference:

- *Monte Carlo methods in financial engineering*, Paul Glasserman, Springer Finance (Stochastic modeling and applied probability)

Two dimensional Monte Carlo integrals

Monte Carlo integrals can be used in more than one dimension. Consider the problem to estimate the area of an irregular shape S inside a rectangle of size $L \times H$. Define the function $F(x, y)$

$$f(x,y) = \begin{cases} 1 & \text{when } (x,y) \in S \\ 0 & \text{when } (x,y) \notin S \end{cases}$$

Consider a sample of N random points (x_k, y_k), then

$$\begin{aligned} A(S) &= \int_0^H \int_0^L f(x,y)\,\mathrm{d}x\,\mathrm{d}y \simeq \\ &\simeq \frac{1}{N}\sum_{k=1}^{N} f(x_k, y_k) = \frac{\mathbf{Num}[(x_k, y_k) \in S]}{N} \end{aligned}$$

Convergence of Monte Carlo integrals

Consider a Monte Carlo integral in d dimensions

$$\int_0^{L_1} \mathrm{d}x_1 \int_0^{L_2} \mathrm{d}x_2 f(x_1, x_2, \ldots) = \frac{1}{N} \sum_{k=1}^{N} f(x_1, x_2, \ldots) + \varepsilon$$

By the central limit theorem, regardless of d,

$$\varepsilon \sim \frac{1}{\sqrt{N}} \quad \Longleftrightarrow \quad N \sim \varepsilon^{-2} = 10^6 \tag{18}$$

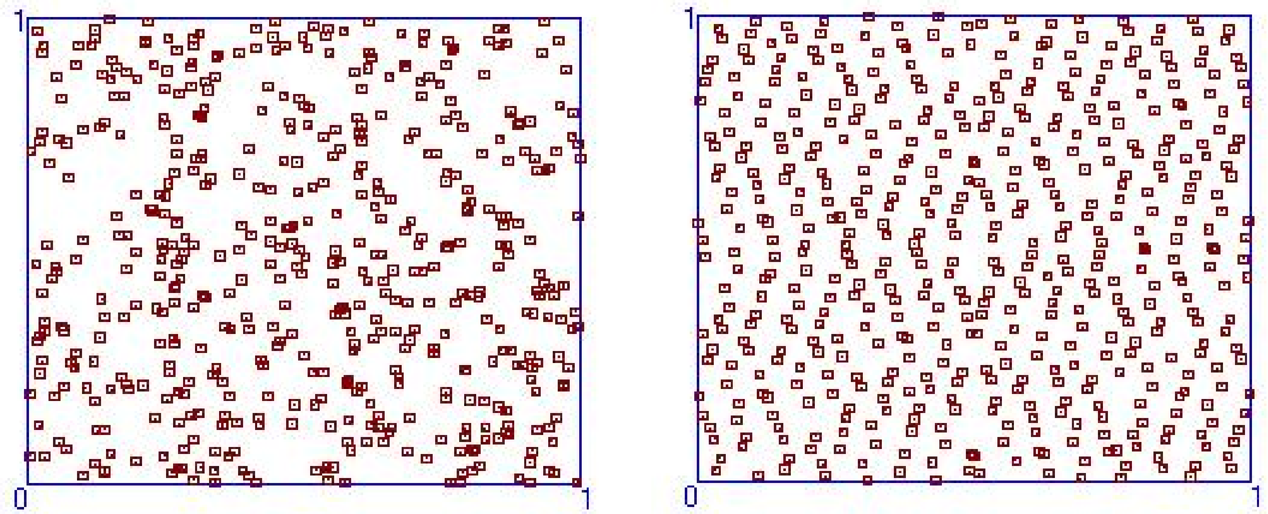
To be compared with the standard trapezoid rule in d dimensions

$$\varepsilon \sim \frac{1}{N^{1/d}} \quad \Longleftrightarrow \quad N \sim \varepsilon^{-d} = 10^{3\,d} \tag{19}$$

The last equalities obtained assuming $\varepsilon = 0.1\%$

Low discrepancy sequences (1/2)

Low-discrepancy sequence can be used instead of random number to compute integrals, to fill up those spaces ...



(courtesy of `http://www.wikipedia.org`)

Low discrepancy sequences (2/2)

Convergence of low-discrepancy sequences is usually better than Monte Carlo simulations:

$$\varepsilon \sim \frac{(\log N)^s}{N} \quad \Rightarrow \quad N \sim 10^{4-5} \quad \text{(Monte Carlo was } 10^6\text{)}$$

- Using *Sobol sequences* is a popular choice
- Only certain N can be chosen
- It is necessary to know in advance the number of simulations
- Different sequences should be used for different dimensions

Summary: Monte Carlo simulations in finance

- Monte Carlo methods are widely used in finance to compute asset prices (sometimes even when not necessary and analytical solutions are available)
- Implicitly or explicitly, performing a Monte Carlo simulations implies some making assumptions on the underlying dynamics
- There is a number of variance-reduction technique to reduce the error (control variates, antithetic, ...)
- Sensitivities to underlying variables need special care
- American-style options with Monte Carlo simulations are possible but very tricky

Questions?

Forward rate models

The dynamic of forward rates, not the instantaneous rate, determines the whole interest-rate structure

Examples

- Black model for Cap pricing: it is assumed that the Libor rates at the reset dates of a Cap have a log-normal distribution. Only few instruments can be priced this way

- Libor market model: similar to the Black model but all the forward rates are considered simultaneously. Forward measures are properly accounted for

Libor market models

A series of models designed to exactly incorporate both the discount factor and the quoted volatilities for Caps and Floors

- Based on the dynamic of the Libor leg of a swap
- There is more than one way to do it
- Has been used in the industry long before its publication (see, e.g., Rebonato)
- First made public by Brace, Gatarek, and Musiela (BGM 1997)
- Prices are computed using Monte Carlo simulations

Forward Libor dynamic

Consider the Libor leg of a swap resetting at dates $T_0, \ldots, T_n$ and paying at dates $T_1, \ldots, T_{n+1}$. For each coupon $j = 1, \ldots, n$ the payoff is proportional to

$$C_j = D(T_{j+1}) \left[\frac{D(T_j)}{D(T_{j+1})} - 1 \right] = D(T_{j+1}) L_j$$

with L_j the j-th forward Libor rate.

In the (forward) Libor market models the market quotes the σ_j as

$$\mathrm{d}L^j = 0 \cdot \mathrm{d}t + \sigma_j \, L_j \, \mathrm{d}W^{j+1} \tag{20}$$

The terminal probability measure

Assumption (20) implies the exact re-pricing of each optionlet separately from the others. In practice we have n distinct numeraires and n equivalent martingales measures.

We will start by rebasing all the Brownian motion to that associated to the longest expiry date. The numeraire is then given by a zero-coupon bond expiring with the payment of the latest cap (or floor).

The corresponding equivalent martinagle measure is called the *terminal probability measure*

Change of measure

For each optionlet we perform a change of measure to the terminal measure (the last Libor rate).

The relation from one period measure to the next is given by

$$\mathrm{d}W_t^j = \mathrm{d}W_t^{j+1} - \frac{\tau_j\,\sigma_j\,L^j}{1+\tau_j\,L_j}\mathrm{d}t \tag{21}$$

with $\tau_j = T_{j+1} - T_j$. Hence recursively, denoting with $Z_t = W_t^{n+1}$,

$$\mathrm{d}L_j = -\sum_{k=j}^{n} \frac{\sigma_k\,\tau_k\,L_k}{1+\tau_k\,L_k}\sigma_j\,L_j\,\mathrm{d}t + \sigma_j\,L_j\,\mathrm{d}Z_t \tag{22}$$

we observe a "drift" in the Libor dynamics

Numerical simulations of LMM (1/3)

The drift presence excludes any analytical computation. We create a Monte Carlo simulation of the *terminal* process Z_t

$$\begin{aligned} Z(T_0) &= 0 \\ Z(T_{j+1}) &= Z(T_j) + \sqrt{\tau_j}\ \varepsilon_j \end{aligned} \tag{23}$$

where ε_j's are normally-distributed independent random numbers.

Since equation,

$$\mathrm{d}x_t = M\,\mathrm{d}t + v\,x_t\,\mathrm{d}Z_t \tag{24}$$

has a solution

$$x(T_{j+1}) = x(T_j)\exp\left\{\left(M - \frac{v^2}{2}\right)\tau_j + v\left[Z(T_{j+1}) - Z(T_j)\right]\right\} \tag{25}$$

Numerical simulations of LMM (2/3)

Equation (22) can be discretized as

$$
\begin{aligned}
L_i(T_{j+1}) \;&=\; L(T_j) \cdot \exp\left\{ -\sigma_i \, L_i(T_j) \sum_{k=i}^{n} \frac{\sigma_k \, \tau_k \, L_k(T_j)}{1 + \tau_k \, L_k(T_j)} \, \tau_j - \frac{\sigma_i^2}{2} \, \tau_j \right\} \cdot \\
&\qquad \cdot \exp\left\{ \sigma_i \left[Z(T_{j+1}) - Z(T_j) \right] \right\}
\end{aligned}
$$

where the last term can be written as

$$
\cdot \exp\left\{ \sigma_i \left[Z(T_{j+1}) - Z(T_j) \right] \right\} = \cdot \exp\left\{ \sigma_i \, \sqrt{\tau_i} \, \varepsilon_j \right\} \qquad (26)
$$

The same random number ε_j should be used for all Libor rates L_i

Numerical simulations of LMM (3/3)

Starting with the Libor rates $L_j(T_0)$'s observed on the Libor curve at the current time, i.e. T_0, we first determine the Libor rates at the next reset date T_1 using (22) and continue in this way until all Libor rates are simulated.

Discounts are computed recursively as

$$D(T_{j+1}) = \frac{D(T_j)}{1 + \tau_j L_j} \tag{27}$$

Note that discounts at intermediate dates should not be interpolated but obtained using intermediate steps

Exercise: numerical LMM simulations for n=2

Simulate a two-period Libor-market model. The simulation dates involved are: T_0 the current time, T_1 the reset time for the first unknown Libor rate L_1, T_2 payment date for the first rate and reset date for L_2, and the final payment date T_3.

Assume σ_1 and σ_2 to be the Caplet volatilities for L_1 and L_2, and $D(T_1)$, $D(T_2)$, and $D(T_3)$ the risk-free discounts observed at T_0.

Consider a simulation according to the terminal measure $Z = W^3$. Compute the price of the path-dependent option with stochastic cash flows C_2 and C_3 at T_2 and T_3, given two independent normally-distributed random numbers ε_1 and ε_2.

Solution of LMM problem (1/2)

First compute the initial forward rates

$$L_1(T_0) = -\frac{1}{\tau_1}\left(\frac{D(T_2)}{D(T_1)} - 1\right) \qquad \text{with } \tau_1 = T_2 - T_1$$

$$L_2(T_0) = -\frac{1}{\tau_2}\left(\frac{D(T_3)}{D(T_2)} - 1\right) \qquad \text{with } \tau_2 = T_3 - T_2$$

Then compute the T_1-simulated Libor rates using ε_1 and ε_2,

$$L_1^{\varepsilon}(T_1) = L(T_0) \cdot \exp\left\{-\frac{\sigma_1^2\,\tau_1\,L_1(T_0)^2}{1+\tau_1\,L_1(T_0)}\tau_0 - \frac{\sigma_1^2}{2}\tau_0 + \sigma_1\sqrt{\tau_0}\,\varepsilon_1\right\}$$

$$L_2^{\varepsilon}(T_1) = L(T_0) \cdot \exp\left\{\sigma_2\sqrt{\tau_0}\,\varepsilon_1\right\}$$

$$L_2^{\varepsilon}(T_2) = L(T_1) \cdot \exp\left\{\sigma_2\sqrt{\tau_1}\,\varepsilon_2\right\}$$

Solution of LMM problem (2/2)

Compute the discount factors on the path, denote $\varepsilon = (\varepsilon_1, \varepsilon_2)$

$$D_\varepsilon(T_2) = \frac{D(T_1)}{1 + \tau_1 \, L_1^\varepsilon(T_1)}$$

$$D_\varepsilon(T_3) = \frac{D_\varepsilon(T_2)}{1 + \tau_2 \, L_2^\varepsilon(T_2)}$$

Finally, given two generic random cash flows at C_2^ε and C_3^ε we can write the option price as

$$PV_\varepsilon \;=\; D(T_3) \sum_\varepsilon P(\varepsilon) \Big\{ C_2^\varepsilon \Big[1 + \tau_1 \, L_1^\varepsilon(T_1) \Big] + C_3^\varepsilon \Big\} \,.$$

The rolling forward deposit numeraire

In LMM simulations for an exotic pricer we need an appropriate numeraire: the *rolling forward deposit*.

Start a T_0 with $M(T_0) = 1$ invest in a deposit, at expiry reinvest everything in another deposit:

$$
\begin{aligned}
M(T_1) &= (1 + \tau L_0) \\
M(T_2|_{T_1}) &= (1 + \tau L_0)(1 + \tau L_1|_{T_1}) \\
M(T_3|_{T_2}) &= (1 + \tau L_0)(1 + \tau L_1|_{T_1})(1 + \tau L_2|_{T_2}) \\
\ldots &= \ldots
\end{aligned}
$$

The discount factor is a stochastic quantity,

$$D_\varepsilon(T_i) = \frac{1}{M(T_i)}$$

Questions?

Advanced numerical simulations

- De-correlation can be introduced between rates
- Usually $\sigma_i(t)$ is a time-dependent function
- Discount rates between nodes are simulated using a *Brownian bridge*
- More than one factor can be used: multi-factor LMM
- Swaption prices are not recovered exactly
- Early prepayments are challenging: e.g., Longstaff-Schwartz method

Swap market models

A series of models designed to exactly incorporate both the discount factor and the quoted volatilities for Swaptions

- Based on the dynamic of the fixed leg of a swap
- There is more than one way to do it
- Just like LMM has been used in the industry long before its publication
- First made public by Jamshidian in 1998 (why is not called the Jamshidian model?)
- Mostly solved using Monte Carlo simulations

Stochastic-volatility models

- Developed to account for the smiled volatilities
- There are flavors suitable for both Libor-market and swap-market models
- Cannot be easily calibrated on observed smiles: calibration needs to be done almost manually
- A large number of assumptions must be made on the correlation between all volatilities
- Used only by banks with large quant groups
- Computationally expensive

Questions?

References

- *Efficient methods for valuing interest rate derivatives*, Antoon Pelsser, Springer Finance
- *The Complete Guide to Option Pricing Formulas*, Espen Gaarder Haug, Mc Graw Hill (from first edition)
- *Monte Carlo methods in financial engineering*, Paul Glasserman, Springer Finance (Stochastic modeling and applied probability)