Average-maturity Model for Asset Backed Securities

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Version 1.2

March 2009*

Abstract

We describe an average-maturity method to model prepayment for asset-backed securities for which it may or may not be present an expiration date.

1 Pricing of Asset-Backed Securities

Asset-backed securities are a popular investor choiche to diversify portfolios. For an introduction to asset-backed securities refer to reference [1].

Consider the following, very simplified version, of an asset-backed security based on stochastic repayments such as credit-card payments. A number m of individuals take out a loan, for example a credit-card debt, for an amount of L each. Let us assume that the amount L is the same for all the individuals and can only be repaid in full. For any given payment period, for example a month, each investor can decide to either repay the loan in full or to pay an interest r on the amount borrowed. Now consider the investor that lends out all the loans. From her point of view she issued a bond with notional $N = m\,L$ were the redemption is paid out at random times. In other words, she is holding a ABS. The goal of this document is to derive a simple prepayment model in order to compute the net present value of this ABS. In order to simplify the discussion further, in the following we will assume a large, possibly infinitely large, m and a small loan amount L, possibly infinitesimal, so that $N = m\,L$ is finite. We will also assume that all the credit risk associated with the possibility of any issuer to not be able to pay back the loan will be included in the discount factor.

1.1 Unlimited time horizon

The first version of ABS that we consider is one where the investors have no time limit to repay their debt L. Since each individual has borrowed the same amount as the others, the amount of notional paid back is proportional to the amount of people that, at a certain time, still need to pay back their loan. Denote, therefore, at any time t the notional that is left to be paid $n(t)$. At the beginning, i.e. at time $t = 0$, we have the initial condition on N

$$n(0) = N. \tag{1}$$

Consider now a generic time t, assuming that all the individuals that took out the loans take their decision of prepayment independently from each other. Also, suppose that a small interval of time

*Latest review January 27, 2011

dt has elapsed, then the amount of outstanding notional dn paid back in this small time frame is proportional to the current outstanding $n(t)$. In other words, the number of people deciding to pay back their loan is proportional, with proportionality constant μ, to the number of people that did not pay back yet. Therefore,

$$dn = \mu\ n\ dt\,, \tag{2}$$

so that we can write that the new amount of outstanding debt is equal to the previous amount minus the prepayment

$$n(t+dt) = n(t) + dn = n(t) - \mu\ n(t)\ dt \tag{3}$$

from which it follows the differential equation

$$\frac{dn}{dt} = -\mu\ n. \tag{4}$$

It is convenient to define the parameter τ, with the dimension of time, as

$$\tau = \frac{1}{\mu} \tag{5}$$

so that expression (4) becomes

$$\frac{dn}{dt} = -\frac{n}{\tau}. \tag{6}$$

The solution of which that satisfies the initial condition (1) is easily obtained as

$$n(t) = N\ e^{-\frac{t}{\tau}}\ . \tag{7}$$

Let us now compute the individual average time of repayment

$$\bar{t} = \int_0^\infty t\ \frac{1}{N}\ (-dn) = \int_0^\infty t\ \frac{1}{N}\ \frac{n}{\tau}\ dt = \int_0^\infty t\ \frac{1}{N}\ \frac{1}{\tau} N\ e^{-\frac{t}{\tau}}\ dt \tag{8}$$

$$= \int_0^\infty \frac{t}{\tau}\ e^{-\frac{t}{\tau}}\ dt = \int_0^\infty \frac{t}{\tau}\ e^{-\frac{t}{\tau}}\ \frac{dt}{\tau}\ \tau = \ \tau \int_0^\infty \xi\ e^{-\xi}\ d\xi = \tau. \tag{9}$$

Hence, τ turns out to be exactly the average prepayment time.

1.2 Finite maturity horizon

Now consider the case where each individual needs to repay his loan before a certain maturity T. This case is similar to the previous one but with an additional time scale T. To model dn one would like to come up with an expression similar to (2), however introducing the absolute need to prepay before time T. The proportionality constant μ should, therefore, now increase as time t approaches to T. This can be achieved, for example, using the expression

$$\mu = \frac{\frac{1}{\tau}}{1 - \frac{t}{T}}\ . \tag{10}$$

Note that the parameter $\frac{1}{\tau}$ at the numerator gives the right time scale to μ, so that for small times compared with T we recover the derivation of sub section 1.1,

$$\lim_{t\to 0} \mu = \frac{1}{\tau}, \tag{11}$$

which is consistent with (5). The denominator of μ, $1-\dfrac{t}{T}$, is such that

$$\lim_{t\to T^-} \mu = +\infty \tag{12}$$

so that all the prepayments will be made before time T with probability one.

The differential equation associated to $n(T)$ is therefore

$$\frac{dn}{dt} = -\frac{\dfrac{1}{\tau}}{1-\dfrac{t}{T}}\; n. \tag{13}$$

from which

$$-\tau\frac{dn}{n} = \frac{1}{1-\dfrac{t}{T}}\; dt, \tag{14}$$

and

$$-\tau\int_{n(0)}^{n(t)} \frac{dn'}{n'} = \int_0^t \frac{1}{1-\dfrac{t'}{T}}\; dt', \tag{15}$$

substituting $\xi = 1-\dfrac{t'}{T}$, if and only if $t' = T(1-\xi)$, so that,

$$-\tau\log\frac{n(t)}{N} = \int_0^t \frac{1}{1-\dfrac{t'}{T}}\; dt' = -T\int_1^{1-\frac{t}{T}} \frac{1}{\xi}\; d\xi = -T\;\log\left(1-\frac{t}{T}\right), \tag{16}$$

and, finally,

$$n(t) = \; N\left(1-\frac{t}{T}\right)^{\dfrac{T}{\tau}}. \tag{17}$$

The average time of repayment can be computed as,

$$\bar{t} = \int_0^T t\;\frac{1}{N}\;(-dn) = \frac{1}{N}\;\int_0^T t\;\frac{dn}{dt}\;dt = \frac{1}{N}\;\int_0^T n\;dt\;, \tag{18}$$

where, in the last equality, we integrated by parts and used the fact that $n(T)=0$. Substituting equation (17) one obtains

$$\bar{t} = \frac{1}{N}\;\int_0^T N\left(1-\frac{t}{T}\right)^{\dfrac{T}{\tau}}\;dt = \;\int_0^T\left(1-\frac{t}{T}\right)^{\dfrac{T}{\tau}}\;dt\;=\;(-T)\int_{\xi=1}^{\xi=0}\xi^{\dfrac{T}{\tau}}\;d\xi = \tag{19}$$

$$= \;\frac{T}{\dfrac{T}{\tau}+1}\left[\xi^{\dfrac{T}{\tau}+1}\right]_{\xi=0}^{\xi=1} = \frac{T}{\dfrac{T}{\tau}+1} = \frac{\tau}{1+\dfrac{\tau}{T}}, \tag{20}$$

where, again, we used the substitution $\xi = 1 - \frac{t}{T}$, and assumed

$$\frac{T}{\tau} + 1 > 0, \tag{21}$$

In summary we have the expression,

$$\bar{t} = \frac{\tau}{1 + \frac{\tau}{T}}, \tag{22}$$

that, given an average time $\bar{t}$ and a maturity date T, can be solved for τ to give,

$$\tau = \frac{\bar{t}}{1 - \frac{\bar{t}}{T}}, \tag{23}$$

Notice that in the limit of $\tau \ll T$ we have exactly the same result as the case with an unlimited time horizon, $\bar{t} = \tau$.

1.3 Conclusions

Using simple arguments on the individual prepayments we were able to deduce a form for the outstanding notional as a function of time to be, in the case of unlimited time horizon

$$n(t) = N\ e^{-\frac{t}{\tau}}, \tag{24}$$

where τ is the expected time of prepayment and N is the initial notional. In case, however, in which all the prepayments have to be made before a time T, the distribution of outstanding notional can be model to be

$$n(t) = N\left(1 - \frac{t}{T}\right)^{\frac{T}{\tau}}, \tag{25}$$

where τ is related to $\bar{t}$ and T by equation (22).

Given an ABS one can use one of the two prepayment models described to create an amortizing bond with a notional paid according to $n(t)$. This amortizing bond can be priced using the standard procedure to compute a value for the given asset-backed security.

References

[1] Wikipedia. Coherent risk measure. `http://en.wikipedia.org/wiki/Asset-backed_security`.