Foundations of the Statpro Simulation Model

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Version 1.1

October 2007*

Abstract

This is the first in a series of papers describing the *StatPro simulation model*. In this paper we describe the basic building blocks of the whole simulation model: given a financial instrument whose price depends on some basic risk factors through a pricing function, it is possible to obtain the expected distribution of asset value from the expected distribution of the underlying risk factors. Therefore, a proper modeling of each possible risk factor is an essential feature of the model. The model here described has been implemented by StatPro in its SRM product and has been proven to be reliable and robust.

1 Introduction

Almost all approaches in risk management attempt to model the future expected distribution of price returns for one asset, or a portfolio of assets, and extract from that distribution some risk indicators. For example, the widely-used variance/covariance approach(see, for example, reference [4]), assumes that the expected distribution of any asset is Gaussian and extracts the relevant risk measures from the one parameter that describes the distribution width, i.e. the volatility. Other models, such as historical simulation(see reference [2]), do not make any analytical assumption on the expected distribution of returns and work instead with *discrete* distributions. The StatPro model follows this second approach.

This paper will begin by describing the procedure to compute scenarios for a single stock, will then generalize the procedure to a portfolio of stocks, and, eventually, to a generic instrument for which a pricing function is known.

2 The expected distribution for a simple asset

Before describing the more complicated procedure used to compute the expected distribution of a portfolio it is necessary to understand very well the same procedure for a simple financial instrument. In the present paper we use interchangeably the words: financial instrument, instrument, product, or asset. An instrument will be considered simple if it is completely independent from any other asset in the market and, therefore, cannot be expressed as a function of assets. On the other hand a complex product is one that can be expressed as a function of one or more simple assets. For example, a stock is a simple product while an option on the same stock is a complex product. This section describes how to obtain expected scenarios for simple products. First the two different types

*Latest review January 27, 2011

of scenarios are described: multiplicative and additive. Subsection 2.1 will show the computation of scenarios for stocks and similar assets starting from the historical prices. The same procedure then will be generalized to other types of products and risk factors.

2.1 Basic scenario types

Given a certain asset, let's call P_t its price at the current time t. Evaluating an asset risk means trying to estimate the possible future values for that asset. The goal of risk management is to define a discrete distribution of N possible price scenarios P_i, $i = 1, \ldots, N$, for the given asset price P_T at time T in the future with $T > t$. For all practical purposes one can think of the time t as today and the time T as the next business day. However, whenever possible, the discussion will be kept as general as possible so that it can easily be extended to periods longer than one day. It is important to stress the difference between P_T and the P_i's: the price P_T is not known at time t and will be only known at time T. On the other hand, the prices P_i's are known at time t and are only useful to determine the statistics, i.e. the probability distribution, of P_T. The prices P_i's do not constitute any attempt to forecast the precise value of P_T and therefore do not make sense taken singularly. However, even though it is unlikely that the value of P_T matches exactly that of P_i for any i, a sensible risk-management model requires each P_i to be a reasonable guess for P_T. In other words, the collection of all P_i's gives the expected distribution of the asset price P_T. In the following the terms statistics, simulated scenarios, and expected distribution always refer to the collection of P_i, for i=1, ..., N. We will denote the collection of all P_i, for i=1, ..., N, simply with $\{P_i\}$.

Multiplicative scenarios Given a certain asset, or financial instrument, for which a positive price is always expected, for example a stock or a bond, both P_t and P_T should be positive, the scenarios $\{P_i\}$ can be built generically from N positive numbers S_i as,

$$P_i = P_t \, S_i \qquad\qquad \text{for} \quad i = 1 \ldots N \,. \tag{1}$$

The numbers S_i's constitute the expected distribution of returns and are the bases for the multiplicative scenarios P_i because it is sufficient to multiply them by P_t to obtain P_i. Notice that the numbers S_i are pure numbers and do not carry, for example, a currency as the prices P_i's.

Additive scenarios In certain cases, for example a swap or other over-the-counter instruments, it is not possible to ensure that the price of a certain asset is or will be strictly positive during its life. In these cases it is more useful to express the future scenarios $\{P_i\}$ as,

$$P_i = P_t + S_i^+ \qquad\qquad \text{for} \quad i = 1 \ldots N \,, \tag{2}$$

the sum of the current price P_t and the additive base scenarios $\{S_i^+\}$. It should be noted that, differently from the simulated returns $\{S_i\}$, the base scenarios $\{S_i^+\}$ do carry the currency or, more in general, the dimension of the original asset. From now on, unless otherwise specified, whenever the index i appears in an expression it is understood to run from 1 to N.

Different risk-management systems make different assumptions in order to model the scenarios P_i's. The goal of the present paper is to describe the building blocks of the **StatPro simulation model** for simulating the expected distribution $\{P_i\}$. StatPro's risk-management platform was inspired by the historical-simulation(see, e.g., reference [2]) approach and was improved by a proprietary methodology for taking into account the risk coming from a number of different risk factors. In particular, the StatPro model has a unique way to deal with the dependence upon credit risk. We refer to a companion paper, see reference [1], for more details on the scenario computations for credit risk.

2.2 Expected distribution for a stock

We will begin with the computation of the simulated distribution for a simple stock and then we will extend it to more complex assets. For simplicity, it will be assumed that the horizon period for the evaluation of risk is one day, so that t refers to today and T to tomorrow. The expected distribution of prices for a stock, can be expressed using definition (1) for an appropriate choice of the expected distribution of returns $\{S_i\}$. In practice, most risk-management models adopt either of the two following approaches:

1. Assume that the S_i's follow a well-known analytical continuous distribution, usually the normal or the log-normal distribution
2. Use the historical time series of returns up to a certain date in the past to model the future distribution

The first assumption is made by a number of models such as the Monte Carlo simulation method or the variance/covariance method. The second assumption is made by the historical model, historical-simulation model, and the StatPro simulation model here described. In short it is assumed that the simulated returns are distributed like the historical ones. More precisely, the scenario distribution for a stock is based on the *stationary assumption*: it is assumed that **the return of the stock from today to tomorrow is distributed like the historical frequency of the daily returns for the same stock during a given past period** (e.g. two years). More explicitly, if the historical closing values of the product for day i are denoted by P_i^{hist}, the scenarios S_i are defined as

$$S_i = \frac{P_i^{\text{hist}}}{P_{i-1}^{\text{hist}}} \,. \tag{3}$$

The return scenarios here defined should be used in conjunction with expression (1) to define the price scenarios.

In order to provide for simulations that adapt more quickly to the market, it is also possible to adjust the distribution scenarios in order to reflect the most recent volatility such as the method suggested by reference [3].

Newly or recently issued stocks Sometimes it happens that the stock for which the scenarios are being evaluated does not have enough historical fixings to compute all the needed scenarios as in expression (3). In this case a peer stock is considered, i.e. another stock that has a very similar risk profile to the given one. The old scenarios that are missing are borrowed from the peer stock.

The support desk at StatPro assigns a peer to each stock, or any other instrument, that does not have at least two years of history for daily fixings. Clearly, as time goes by the newly issued assets gets more and more historical fixings and, eventually, the peer stocks are no longer necessary.

2.3 Extension to other asset classes

We show hereafter how the historical scenarios previously defined can be generalized to other asset classes.

Extension of the method to funds and indices The historical scenarios computed by definition (3) for the case of stocks, can be used to compute the scenarios for other asset classes. Specifically, StatPro uses this method for the following asset classes:

- Stocks and equivalent instruments, e.g. ADR(American depository receipt). Without entering in the discussion on if and how the price of a stock can be decomposed into different factors, it is assumed that the only observable on the market is the stock price.

- Funds and ETFs (exchanged-traded funds). These assets will be modelled like a stock only when the user considers the fund as a single unit to be bought or sold as a whole. See section 3 for funds viewed as portfolios.
- Indices are handled similarly to stocks when the index is seen as a non-decomposable unit, otherwise as a portfolio (again, see section 3 for more details).

Foreign-exchange rates As shown later in section 3, the scenarios for a foreign-exchange rate are necessary in the computation of portfolio scenarios. Even though a foreign-exchange rate is not a proper asset, its scenarios can be computed using a definition similar to (1). Given two different currencies, let F_t be the last fixing for the exchange rate, and let F_T be the future rate for which the N scenarios are needed. Then define,

$$F_i = S_i\, F_t, \tag{4}$$

and the historical return scenarios S_i by,

$$S_i = \frac{F_i^{\text{hist}}}{F_{i-1}^{\text{hist}}}\,, \tag{5}$$

where F_i^{hist} is the fixing of the exchange rate at date i in the past.

Extension of the method to additive scenarios Expression (3) can be used to obtain the scenarios S_i necessary for the multiplicative scenarios defined in (1). Using the same approach it is possible to obtain an expression for the additive scenarios described in equation (2),

$$S_i^{+} = P_i^{\text{hist}} - P_{i-1}^{\text{hist}}\,. \tag{6}$$

This type of scenarios is used when it does not make sense to compute the ratio of two consecutive prices because they could be close to zero, negative, or simply, because it makes more sense to consider differences instead of ratios. For example formula (6) is used in computing the scenarios for credit spreads or interest rates.

2.4 Computation of risk measures at asset level

In this paragraph we compute two different risk measures for a single stock: value at risk(VaR for short) and expected shortfall(ES for short) or conditional VaR. As an example consider a certain stock a, with current price P_t^a=3.50, for which there are the latest 501 daily closing prices, i.e. about two years of history. Table 1 shows an example of computation for the return distribution $\{S_i\}$ using expression (3) and $\{P_i\}$ using definition (1). Once the scenarios P_i^a are available, it is in principle possible to compute any risk measure. Here, for brevity, only the computation of value at risk and expected shortfall will be shown.

Value at risk Given a certain percentile $p^{\%}$ the value at risk of an asset is defined as the *Maximum possible loss in the value of an asset within a given time span and at a given confidence level* $p^{\%}$. For example, for a percentile $p^{\%}$= 99 % to compute VaR 99 %, we order the P_i's from the smallest to the largest. Since the distribution of returns is made of N=500 observations, to obtain the loss with a 99 % confidence level we need to consider the lowest loss of the worst $n = (1 - p^{\%})N$=5 observations. In Table 2 we also compute the percentage loss $\lambda_i^{\%}$ for each scenario i,

$$\lambda_i^{\%} = 1 - \frac{P_i^a}{P_t^a}. \tag{7}$$

	$i=0$	$i=1$	$i=2$	...	$i=498$	$i=499$	$i=500$
P_i^{hist}	2.81	2.78	2.95		3.61	3.33	P_t^a=3.50
S_i	undefined	0.98932	1.06115		0.93282	0.92244	1.05105
P_i	undefined	3.4626	3.7140		3.0823	3.2285	3.6787

Table 1: Computation of return scenarios $S_i = P_i^{\text{hist}}/P_{i-1}^{\text{hist}}$ and price scenarios $P_i = S_i\, P_t$ for stock a. For brevity, scenarios for i =3, . . ., 497 have been ommitted.

	$i=435$	$i=71$	$i=267$	$i=29$	$i=498$	$i=114$	...
ordinal	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	...
P_i	2.8735	2.9426	2.9840	3.0536	3.0823	3.0901	...
$\lambda_i^{\%}$	17.90%	15.93%	14.74%	12.75%	11.93%	11.71%	...

Table 2: Price scenarios P_i in increasing orderd from the smallest to the largest. It is also shown the percentage loss $\lambda_i^{\%}$ computed as in equation (7).

The best observation of the worst 5 scenarios is the VaR, i.e. the scenario tagged with i=498 and corresponds to P_{498}^a=3.0823, $\lambda_{498}^{\%}$=11.93 %, therefore,

$$VaR^{99\,\%}(a) = 1 - \frac{P_{498}^a}{P_t^a} = 1 - \frac{3.0823}{3.5} = 11.93\,\%. \tag{8}$$

Expected shortfall The expected shortfall for a certain percentile $p^{\%}$, for example $p^{\%}$=99%, is the average loss expected in all but the best 99 % of the cases. In other words it is obtained by averaging the observations comprised within the worst 1 % percentile. In the example of Table 2, the expected shortfall for stock a is given by the average of the worst five values

$$ES^{99\%}(a) = \frac{\lambda_{435} + \lambda_{71} + \lambda_{267} + \lambda_{29} + \lambda_{498}}{5} = 14.65\%. \tag{9}$$

Notice that, for the same value of the percentile, the expected shortfall is always larger than the corresponding value at risk.

Once the expected distribution $\{P_i^a\}$ is computed, it is easy to compute other measures like, for example, volatility or potential upside.

Note In the computation of value at risk and expected shortfall, in general, the index $n = (1 - p^{\%})N$ is not an integer. In this case a value at risk and expected shortfall are computed for n^-, the closest integer smaller than n, and for n^+, the closest integer higher than n. The value of the sought risk figure is then interpolated between the two computed values.

3 Scenario distribution at portfolio level

An essential feature of a risk-management model is the ability to compute the risk measures of a portfolio of assets. This section begins by showing the scenario computation for a small portfolio of three stocks giving also a an example of computation. The same procedure is then generalized to the scenario computation of a generic portfolio with any number of assets of all types.

3.1 Scenarios for a portfolio of three stocks

Let's assume for simplicity to own a portfolio made of three stocks, a, b, and c not necessary all in the same currency. Suppose that we own each of the stocks in quantities, respectively, Q^a, Q^b, Q^c, and that their current price is P_t^a, P_t^b, P_t^c, so that the total portfolio value at time t is given by,

$$\Pi_t = \frac{Q^a\,P_t^a}{F_t^a} + \frac{Q^b\,P_t^b}{F_t^b} + \frac{P_t^c\,Q^c}{F_t^c}\,, \tag{10}$$

where F_t^a, F_t^b, F_t^c, are, respectively, the currency-exchange rates, at time t, of the currencies of stocks a, b, c with respect to the portfolio currency.

If the portfolio composition does not change from time t to time T then the portfolio future value will be given by,

$$\Pi_T = \frac{Q^a\,P_T^a}{F_T^a} + \frac{Q^b\,P_T^b}{F_T^b} + \frac{P_T^c\,Q^c}{F_T^c}\,. \tag{11}$$

The problem with this expression is that the quantities P_T^a, P_T^b, P_T^c, and F_T^a, F_T^b, F_T^c are not known at time t. However, the expected distributions $\{P_i^a\}$, $\{P_i^b\}$, $\{P_i^c\}$ for the stock prices can be computed by expression (1) and, similarly, the scenarios $\{F_i^a\}$, $\{F_i^b\}$, $\{F_i^c\}$ for the foreign-exchange rates are given by definition (4). Hence, conditionally to the realization of scenario i, the portfolio value at time T will be,

$$\Pi_i = \frac{Q^a\,P_i^a}{F_i^a} + \frac{Q^b\,P_i^b}{F_i^b} + \frac{P_i^c\,Q^c}{F_i^c}\,. \tag{12}$$

Therefore, the expected distribution for the portfolio value is given by the collection of all Π_i for i from 1 to N, i.e. $\{\Pi_i\}$.

Interestingly, notice how expression (12) is very general and is not tied to the historical simulation scenarios defined by (3).

Example: computation of portfolio scenarios As a detailed example consider a portfolio valued in dollars composed of three stocks: the first one, stock a, is in dollars and is held in quantity Q^a=10 shares, the second one, stock b, is in Japanese Yens and it is held in quality Q^b=50 shares, the last one, stock c, is in British Pounds and is held in quantity Q^c=100 shares. Given that their current stock prices are: P_t^a =91.34 \$, P_t^b=3,728 ¥, P_t^c=1.89 £; the currency-exchange rates are F_t^a=1, F_t^b=149.54¥/\$ for EUR-JPY, F_t^c=0.6766 £/\$ for EUR-GBP; the current portfolio value is 4,075.32 \$. Table 3 gives an example of possible scenarios for the stock prices P_i^a, P_i^b, P_i^c, and for the currency-exchange rates F_i^a, F_i^b, F_i^c. Formula (12) allows us to compute the portfolio distribution $\{\Pi_i\}$.

3.2 Portfolio containing any number of assets

The above formula (12) was derived for a portfolio of three stocks with scenarios computed using equation (1). However, it can be easily generalized to a portfolio of any size containing any type of products. In order to do so, consider a portfolio Π evaluated in some reference currency, composed of a number of assets a, b, ..., z, respectively in quantities Q^a, Q^b, ..., Q^z, with current prices P_t^a, P_t^b, ..., P_t^z, in their own currencies, with currency-exchange rates F_t^a, F_t^b, ..., F_t^z. The portfolio current value can be computed as

$$\Pi_t = \frac{Q^a\,P_t^a\,x^a}{F_t^a} + \frac{Q^b\,P_t^b\,x^b}{F_t^b} + \ldots + \frac{P_t^z\,Q^z\,x^z}{F_t^z}\,, \tag{13}$$

where, for each product, e.g. product q, the price factor x^q is the number such that $Q^q\,P_t^q\,x^q$ is the actual amount of money held in the currency of asset q. For example for stocks x=1, for bonds

	current(t)	i=1	i=2	i=3	...	i=500
P^a (\$)	91.34	88.81	81.59	95.44	...	92.34
P^b(¥)	3,728	3,586	3,617	3,723	...	3,725
P^c (£)	12.96	12.82	11.87	13.49	...	11.48
F^a	1.00	1.00	1.00	1.00	...	1.00
F^b (¥/\$)	149.54	153.43	153.74	151.76	...	146.56
F^c (£/\$)	0.6766	0.6612	0.7001	0.7000	...	0.6746
$Q^a\, P^a/F^a$ (\$)	913.40	888.10	815.90	954.40	...	923.40
$Q^b\, P^b/F^b$ (\$)	1,246.46	1,168.61	1,176.34	1,226.61	...	1,270.81
$Q^c\, P^c/F^c$ (\$)	1,915.46	1,938.47	1,695.47	1,927.14	...	1,701.75
Π (\$)	4,075.32	3,995.61	3,687.71	4,108.15	...	3,895.96

Table 3: Computation of the expected distribution for the portfolio value. See the text for more details.

x=1/100, and so on. Suppose that the portfolio composition is held fixed. Then, the portfolio value at time T is given by,

$$\Pi_T = \frac{Q^a\, P^a_T\, x^a}{F^a_T} + \frac{Q^b\, P^b_T\, x^b}{F^b_T} + \ldots + \frac{P^z_T\, Q^z\, x^z}{F^z_T}\,. \tag{14}$$

As in the case of a portfolio of three stocks, the values for the stocks prices and the exchange rates at time T are not known but their expected distributions are. The expected distribution $\{\Pi_i\}$ for the portfolio value Π_T can be evaluated by simply computing the portfolio value for each scenario i. More precisely, the stocks prices and the exchange rates are known conditionally to the realization of scenario i,

$$\Pi_i = \frac{Q^a\, P^a_i\, x^a}{F^a_i} + \frac{Q^b\, P^b_i\, x^b}{F^b_i} + \ldots + \frac{P^z_i\, Q^z\, x^z}{F^z_i}\,, \tag{15}$$

where $\{P^a_i\}$, $\{P^b_i\}$, ..., $\{P^z_i\}$, denote the expected distribution for the portfolio assets and $\{F^a_i\}$, $\{F^b_i\}$, ..., $\{F^z_i\}$ those for their currency-exchange rates.

Once the expected distribution for the portfolio value $\{\Pi_i\}$ has been evaluated, using the techniques described in section 3, it is also possible to compute any risk measure for the portfolio under consideration. If all the assets have positive prices and there are no short position in the portfolio, it is possible to obtain the return scenarios for the portfolio,

$$S^\Pi_i = \frac{\Pi_i}{\Pi_t}\,, \tag{16}$$

from which it follows that

$$\Pi_i = S^\Pi_i\, \Pi_t\,. \tag{17}$$

This formula can be used as the definition of the multiplicative scenarios for a portfolio and should be compared with definition (1).

Notes

- The computation of portfolio scenarios outlined here does not assume in any way that the asset scenarios are multiplicative or additive as explained in section 2

- There is no assumption that the scenarios were generated using the historical method
- The portfolio returns defined in formula (16) should be considered ex-ante returns and have nothing to do with the ex-post return, e.g. used in performance analysis

3.3 Handling of missing scenarios

The computation of portfolio scenarios just outlined relies on all the scenarios to be always defined for all $i = 1, ..., N$. In Table 3, for example, it was implicitly assumed that all scenarios P_i^a, P_i^b, P_i^c, and F_i^a, F_i^b, F_i^c were well defined for every $i = 1, ..., N$. In practice, however, because of week-ends, bank holidays, or simply because of incomplete asset coverage it may happen that some scenarios are not available for some values of i. Hence, it is necessary to define the domain of existence $\mathcal{I}$, i.e. all the valid i's, for which Π_i can be computed by equation (15).

Given a portfolio Π with associated a set of, possibly incomplete, price scenarios $\{P_i^a\}$, $\{P_i^b\}$, ... $\{P_i^z\}$, and a set of, possibly incomplete, foreign exchange scenarios $\{F_i^a\}$, $\{F_i^b\}$, ... $\{F_i^z\}$, consider the set $\mathcal{I}$ of all i's for which there exists at least one scenario among the P_i's or the F_i's: this will be the domain $\mathcal{I}$ of existence of the portfolio scenarios. In other words we want to establish a well defined value of Π_i for all i in $\mathcal{I}$(denoted by $i \in \mathcal{I}$). For each underlying instrument q, define $P_i^q{=}P_t$ every time that P_i^q is not present and $F_i^q{=}F_t$ every time that F_i^q is not present. That is we replace the missing scenarios with the value at time t. In mathematical form, for the price scenarios,

$$P_i^q := \begin{cases} P_i^q & \text{if } P_i^q \text{ is defined} \\ P_t^q & \text{if } P_i^q \text{ is not defined} \end{cases} \qquad \text{for all} \quad i \in \mathcal{I}, \tag{18}$$

and, for the currency-exchange scenarios,

$$F_i^q := \begin{cases} F_i^q & \text{if } F_i^q \text{ is defined} \\ F_t^k & \text{if } F_i^q \text{ is not defined} \end{cases} \qquad \text{for all } i \in \mathcal{I}. \tag{19}$$

With the help of the above two definitions it is possible to compute Π_i from equation (15) for all $i \in \mathcal{I}$.

4 Expected distribution for a zero-coupon bond

Let us consider the computation of the risk distribution for a zero-coupon bond without credit risk, i.e. a bond that does not pay any intermediate coupon but only the notional amount at maturity. Such a bond is traded at a discount from its redemption and its price can be computed bootstrapping an interest-rate term structure using, e.g. an array of deposit and swap rates as shown, for example, in reference [5]. In order to correctly evaluate the price of a bond it is necessary to consider the credit risk associated with its issuer. However, to simplify the discussion that follows, for the time being, the influence of the credit spread on the bond price will be neglected(see reference [1] for scenario computation for a zero-coupon bond in presence of credit risk.)

Consider a string of deposits and swap rates at time nodes t_1, t_2, ..., t_n. Assuming a piecewise-flat-forward interest-rate term structure(see reference [5] for quantitative pricing in the StatPro simulation model) it is possible to bootstrap n instantaneous forward rates $f_t^1, \ldots, f_t^n$. Let also t_B be the time of bond maturity. The bond price at the current time t is given by

$$P_t^B = 100 \exp \Big[-(t_1 - t_S) f_t^1 - (t_2 - t_1) f_t^2 + \dots \\ \dots - (t_j - t_{j-1}) f_t^j - (t_B - t_j) f_t^{j+1} \Big] , \tag{20}$$

where exp is the exponential function, t_S is the time corresponding to the bond settlement[1], and j is the index such that $t_j < t_B \leq t_{j+1}$. It can be noted that the bond price is a function of the current time, the bond maturity, and the current forward rates.

So far, only the bond price was computed at time t, consider the price P_T^B the bond will have at time T. In order to obtain a distribution of prices $\{P_i^B\}$ for the bond price P_T^B, it is possible to proceed in one of the following ways:

1. Consider the historical values of the bond price P_i^{hist} and compute the daily historical scenarios like the stock case as in subsection 2.2
2. Assume a distribution for each of the forward rates $f_T^1, \dots, f_T^n$ and compute from those the simulated bond prices P_i^B's

The first method is much simpler to implement since it requires only the historical values of the past bond prices. However, it does not account for the changing of the bond structure during its life. The scenarios generated in this way are inconsistent with each other as they refer to bond prices with different times to maturity. Furthermore, this procedure cannot be easily generalized to bonds with embedded options such as callable bonds or convertible bonds. In the case of a callable bond, for example, the risk of the bond could be considerably different across callability dates(the procedure outlined later would properly model this behavior). Because of all these problems we shall not proceed any further in the description of first method.

The second method should be implemented as described in the following. The bond price at time T will be given by

$$P_T^B = 100 \exp \Big[-(t_1 - T_S) f_T^1 - (t_2 - t_1) f_T^2 + \dots \\ \dots - (t_j - t_{j-1}) f_T^j - (t_B - t_j) f_T^{j+1} \Big] , \tag{21}$$

where T_S is the time corresponding to the bond settlement date [2] for a bond sold at time T. Here there is a problem: at the current time t the values of the forward rate f_T^1 are not known exactly.[3]

Consider, therefore, the simulated scenarios for the forward interest rates. Each forward rates f^k is considered as a primary financial variable, or a risk factor for the bond price. Since the forward rates can arbitrarily become close to zero, their scenarios f_i^k are modelled as additive scenarios, see definition (2), i.e.,

$$f_i^k = \Phi_i^k + f_t^k \qquad \text{for} \ \ i = 1 \dots N \, , \text{and} \ \ k = 1 \dots n \, , \tag{22}$$

for some choice of the spread scenarios $\{\Phi_i^k\}$.

The historical-simulation scenarios Φ_i^k for the forward rates are computed using definition (6), i.e.,

$$\Phi_i^k = f_i^{k,\,\text{hist}} - f_{i-1}^{k,\,\text{hist}} \qquad \text{for} \ \ i = 1 \dots N \, , \text{and} \ \ k = 1 \dots n \, , \tag{23}$$

[1] Notice that t_S is really a function of t: $t_S(t)$.
[2] More precisely, $T_S = t_s(T)$.
[3] Note, however, that it is possible to compute their expected value in the risk-neutral measure.

where $f_i^{k,\text{ hist}}$ is the k-th forward rate computed from the bootstrapping of the interest-rate curve at date i.

At this point it is possible to compute the bond price at time T conditional to the realization of scenario i. The simulated bond price P_i^B can only be given by non-arbitrage formula (21) with f_T^k replaced by f_i^k, i.e.,

$$P_i^B = 100 \exp\Big[-(t_1 - T_S) f_i^1 - (t_2 - t_1) f_i^2 + \dots \\ \dots - (t_j - t_{j-1}) f_i^j - (t_B - t_j) f_i^{j+1} \Big]. \tag{24}$$

Given the expected distribution $\{P_i^B\}$, it is possible to compute value at risk or expected shortfall, both described in subsection 2.4, or any other risk measures that computable on a discrete distribution.

Note

- The simulated bond prices P_i^B's do not have the shortcomings that the historical prices have. In particular, the duration of each of the bond P_i^B is the same as that of P_T^B
- The technique described above can be applied to newly, or recently, issued bonds without the need of a peer asset

5 Scenarios for complex financial instrument

This section describes how to compute the simulation scenarios for a large number of asset classes. The type of instruments we are interested in are all those products for which it is possible to compute the price, for example using an arbitrage-free pricing function; they include

- Bonds and notes: coupon bonds, floaters, reverse floaters, callable bond, convertible bonds, and so on
- Options both plain/vanilla and exotic of all types
- Futures, such as index futures, stock futures, bond futures, interest-rate futures, and so on
- All type of certificates
- Structured bonds containing fixed and/or floating payments with/without embedded options of all types
- Over-the-counter products such as interest-rate swaps, inflation swaps, credit default swaps, convertible debt obligations, and so on
- Other types of exotic or hybrid products

To obtain the scenario distribution for all these assets we generalize the method described in previous sections for zero-coupon bonds and portfolios.

The procedure to compute the scenarios for a portfolio, as shown in section 3, and the one for a zero-coupon bond, as illustrated in section 4, may seem very different at first but share many similarities as shown below.

The portfolio value in equation (13) can be re-written as

$$\Pi_t = f_\Pi \left(\text{parameters}; P_t^a, \ldots, P_t^z, F_t^a, \ldots, F_t^z\right) , \tag{25}$$

where f_Π is a function of some parameters, in this case Q^a, ...,Q^z, x^a, ...,x^z, that do not change from t to T; and some risk factors, in this case P_t^a, …, P_t^z, F_t^a,…,F_t^z, that affect the portfolio price and could potentially change during the same time period. Formulated in this way the scenario formula (15) becomes

$$\Pi_i = f_\Pi \left(\text{parameters}; P_i^a, \ldots, P_i^z, F_i^a, \ldots, F_i^z\right) . \tag{26}$$

Similarly, the value of a zero-coupon bond, as computed in equation (20), can be written in functional form as

$$P_t^B = f_B(t_S, t_B;\, f_t^1,\, \ldots,\, f_t^n) , \tag{27}$$

and its scenarios in equation (24) as

$$P_i^B = f_B(T_S, t_B;\, f_i^1,\, \ldots,\, f_i^n) . \tag{28}$$

Written in this form the similarities between the computation of scenarios for a portfolio and that for a zero-coupon bond are evident.

To generalize this procedure to a generic asset, consider a financial instrument π, e.g. a portfolio, a zero-coupon bond, or another product, for which it is possible to compute the price at time t as a function of n risk factors r_t^1, ..., r_t^n; possibly depending on K parameters X^1, ...,X^K, i.e. in mathematical terms,

$$\pi_t = f_\pi \left(t;\, X^1, \ldots, X^K;\, r_t^1, \ldots, r_t^n\right) . \tag{29}$$

Note that some of the risk factors could themselves be vectors such as the case of the instantaneous forward rates that are be considered as a single risk factor: the interest rate curve. Consider now the market price of π at the future time T:

$$\pi_T = f_\pi \left(T;\, X^1, \ldots, X^K;\, r_T^1, \ldots, r_T^n\right) . \tag{30}$$

Conditional to the realization of the given scenario i, each risk factor r_T^k assumes the value r_i^k. By definition, the parameters $X^1, \ldots, X^K$ are left unchanged from their value at time t. Therefore, the instrument price π_T is computable using the function f_π and it is equal to,

$$\pi_i = f_\pi \left(T;\, X^1, \ldots, X^K;\, r_i^1, \ldots, r_i^n\right) . \tag{31}$$

Hence, the collection $\{\pi_i\}$ just defined is the expected distribution for π_T.

Note It is possible that the computation of the asset values using f_π involves a Monte Carlo simulation of some sort that will change the value of the underlying risk factors. One should not confuse the Monte Carlo scenarios obtained in the *risk-neutral measure* with the simulated scenarios r_i^k of the *real-world measure.*

The scenarios $\{\pi_i\}$ so computed in equation (31) can be used to compute value at risk, expected shortfall, or other risk figures as described in subsection 2.4. Also, these scenarios can be combined with other assets, for which the scenarios are also known, to compute the scenarios of a containing portfolio as described in section 3.

5.1 Missing scenarios

As in the case of the portfolio scenarios, described in subsection 3.3, here there is the problem that, in general, some scenarios for some risk factors may not be defined on certain dates. This, again, could be due to week-ends, bank holidays, or incomplete asset coverage. As in the portfolio case, the scenarios for the instrument under consideration will be defined only for all those i's($i \in \mathcal{I}$) for which there exists at least one scenario r_i^k for any k. For those i's we re-define r_i^kin the following way:

$$r_i^k := \begin{cases} r_i^k & \text{if } r_i^k \text{ is defined} \\ r_t^k & \text{if } r_i^k \text{ is not defined} \end{cases} \quad \text{for all } i \in \mathcal{I}. \tag{32}$$

Therefore, equation (31) can now be used to properly compute all π_i's for all $i \in \mathcal{I}$.

5.2 Example: scenarios of a vanilla option

Another example of scenario computation is the vanilla option on a stock. For a vanilla option on a stock, using the usual no arbitrage arguments(see, for example, reference [1]), the option price is given by the well-known Black-Scholes formula. The risk factors, i.e. the financial variables from which we expect risk to come from, are the underlying stock price P, the relevant forward rate r, and the implied stock volatility σ. The strike price K and the maturity time t_m, are also input parameters of the pricing formula, but they cannot be considered risk factors as they would remain constant in time. Given the current value P_t, r_t, and σ_t, for the risk factors, the option price is given by the Black-Scholes formula,

$$\begin{aligned} \omega_t \;=\; & P_t \mathcal{N}\left(\frac{\log\left(P_t/K\right) + \left(r_t + \frac{1}{2}\sigma_t^2\right)(t_m - t)}{\sigma_t \sqrt{(t_m - t)}}\right) + \\ & - K\, e^{-r_t(t-t_m)} \mathcal{N}\left(\frac{\log\left(P_t/K\right) + \left(r_t - \frac{1}{2}\sigma_t^2\right)(t_m - t)}{\sigma_t \sqrt{(t_m - t)}}\right) . \end{aligned} \tag{33}$$

This should be considered as the equivalent of equation (29) for the option price. Using equation (29) the computation of scenarios ω_i's is straightforward and are given by,

$$\begin{aligned} \omega_i \;=\; & P_i \mathcal{N}\left(\frac{\log\left(P_i/K\right) + \left(r_i + \frac{1}{2}\sigma_i^2\right)(T - t_m)}{\sigma_i \sqrt{(T - t_m)}}\right) + \\ & - K\, e^{-r_i(T-t_m)} \mathcal{N}\left(\frac{\log\left(P_i/K\right) + \left(r_i - \frac{1}{2}\sigma_i^2\right)(t_m - T)}{\sigma_i \sqrt{(t_m - T)}}\right) , \end{aligned} \tag{34}$$

where $\{P_i\}$ and $\{r_i\}$ are the scenarios, respectively, for the underling stock price and for the relevant forward rate as computed, respectively, in section 2 and 4. The scenarios for the underlying volatility σ_i, can be computed as shown in the following paragraph.

Volatility scenarios The volatility scenarios can be computed similarly to those of as single stock as in section 2.2,

$$\sigma_i = \sigma_t\, S_i^\sigma\,, \tag{35}$$

stock prices	stock volatilities
fund prices	fund volatilities
index prices	index volatilities
currency-exchange rates	currency-exchange volatilities
interest-rate deposits	cap/floor volatilities
interest-rate swaps	swaption volatilities
inflation swaps	hedge-fund indices
credit default swap spreads	base default correlations

Table 4: Risk factors available in the StatPro Risk Management suite (SRM) as the end of year 2006

where,

$$S_i^\sigma = \frac{\sigma_i^{\text{hist}}}{\sigma_{i-1}^{\text{hist}}} \,. \tag{36}$$

The volatility σ_i^{hist} used in the above equation is given by the historical closing values of the quoted implied volatility. When the implied-volatility data are not available one could use instead the annualized historical volatility σ_i^{hist} at date i. Given a fixed window of k days the historical volatility can be computed as the positive square root of the equation

$$\left(\sigma_i^{\text{hist}}\right)^2 = \frac{1}{k-1} \sum_{j=i-k+1}^{i} \xi_j^2 - \frac{1}{k(k-1)} \left(\sum_{j=i-k+1}^{i} \xi_j \right)^2 , \tag{37}$$

where the ξ_i are defined as,

$$\xi_i = \sqrt{252} \, \log \left(\frac{P_i^{\text{hist}}}{P_{i-1}^{\text{hist}}} \right) , \tag{38}$$

for i=1, …, n.

6 Summary and conclusions

Let's summarize the whole procedure of computing the scenarios of a complex instrument. Consider an instrument for which it is possible to compute the arbitrage-free price. This price is therefore a function of a number of risk factors, and a number of parameters. We call the variables that modify the results of a pricing function from time t to time T **risk factors** because these are the factors that determine the expected distribution and, therefore, determine the value of any risk measure. Given the scenarios for the underlying risk factors, it is possible to price the instrument at time T for each scenarios and to generate the expected distribution of prices for the instrument under consideration. Given the expected distribution computed for the instrument it is then possible to either compute a risk measure for it or to compute a risk measure for a portfolio containing the instrument itself.

The generation of each single risk-factor scenarios is therefore of paramount importance for the risk management model as a whole. It was already explained how the risk-factor scenarios are computed for stocks, foreign-exchange rates, forward interest rates, and, finally, for the volatility. The computation of scenarios for credit risk is outlined in a separate paper(see reference [1]).

It should be noted that the variety of risk factors employed by StatPro increases as years go by as new financial asset classes are created and data providers allow for the availability of new market data. The list of risk factors currently available is shown in Table 4. This list can be considered as quite comprehensive for running an accurate assessment of risk for a wide number of asset classes and we believe it represents the current *state of the art*.

References

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