Pricing Simple Credit Derivatives

Marco Marchioro
www.statpro.com
Version 1.4

March 2009*

Abstract

This paper gives an introduction to the pricing of credit derivatives. Default probability is defined and modeled using a piecewise constant hazard rate. Credit default swaps are shown as a first example of credit derivatives. It is then shown how to obtain a default probability term structure that is consistent with market quotes of credit-default swaps. Portfolio credit derivatives are also considered: the loss distribution is computed in both the homogeneous and non-homogeneous cases and is used to compute the price of a collateralized debt obligations. Finally, the generation of simulated scenarios for base correlation is briefly discussed.

1 Introduction

A credit derivative is a contract used to transfer the risk associated to default due to bad credit from one party to another. Credit derivatives became very popular in the recent past past as a distinct asset class. The most popular credit derivatives are credit default swaps (CDS) and collateralized debt obligations (CDO). In this paper we analyze these two credit derivatives and describe how to compute their arbitrage-free value from market data.

A credit default swap provides insurance against the default of a certain company or financial entity. The company is known as the underlying name, or reference, and its default is known as the credit event. In a credit default swap there are two sides: the buyer and the seller of protection. Given the contract notional the buyer of protection makes periodic payments to the seller as a predetermined percentage, known as the CDS spread, of a certain notional value and in return obtains the right to sell a bond issued by the reference entity for its face value if a credit event occurs. Credit default swaps became so popular and trading volumes so high that their quoted spreads have become a measure of the credit worthiness of the underlying names (for a more detailed overview of this see reference [5]). Recently, in order to make credit default swaps more standard, the same fixed spread is applied to many contracts: for example, 100 basis points for investment grade names and 400 basis points for all the others. In this case credit default swaps are quoted as cents over the dollar.

Another popular credit derivative is the collateralized debt obligation, or more precisely the single-tranche collateralized debt obligation. This instrument is an extension of a credit default swap to the losses generated by a basket of underlying names. Given the basket notional a precise loss tranche is singled out: for example all the losses incurred between 6%, the attachment point, and 9%, the detachment point, of the portfolio notional. Again, there is a protection buyer and a protection seller. The protection buyer pays a spread on the tranche notional so that he can receive

*Latest review January 27, 2011

a reimbursement for all the losses incurred in the portfolio after the attachment point and up to the detachment point. Single-tranche CDOs have been standardized and are traded since 2004 on two major indexes: the European iTraxx and the North American CDX. While for credit default swaps the most important factor in pricing is given by the expected probability of default of the underlying name, the value of a CDO tranche is also dependent on the correlation of more than one default to occur in a period of time.

The first part of this paper illustrates single-name credit derivatives: instruments whose payoffs depends on the default of a single issuer. Section 2 describes the concept of probability of default and how it can be expressed in terms of the hazard rate. Section 3 gives more details to the widely used credit default swaps, their pricing, and how it is possible to bootstrap a default probability term structure given a number of CDS quotes on the same issuer. Other popular single-name credit derivatives, such as a corporate bonds and the par asset swaps are introduced in section 5.

The second part of this paper is about portfolio credit derivatives: all those derivative contracts whose payoff depends on defaults of multiple names. In section 6 we explain the modeling of default correlation using a copula model, describing in details the computation of the loss distribution for the homogeneous case, in subsection 6.3, and for the non-homogeneous case in subsection 6.4. Finally, in section 7 we describe in details the pricing of a CDO tranche. The computation of the present value for a single-tranche CDO is outlined in general in subsection 7.2 and in details in subsection 7.3 using the mid-point approximation. We conclude the section outlining the computation of historical-simulation scenarios for base correlation in subsection 7.5.

2 Default probabilities and hazard rates

The pricing of credit derivatives is based on the ability to estimate the risk-neutral probability of default at different future dates. There are different ways to accomplish this task: the simplest one is to assume the default process to be an homogeneous Poisson process with constant hazard h (see, for example, reference [1]). In this case, the probability of default before time t is given by

$$P(\tau_{\text{default}} \leq t) \,=\, P(t) \,=\, 1 - \exp\left(-\int_0^t h\, d\tau\right) \,=\, 1 - \exp\left(-h\, t\right) \,. \tag{1}$$

Note that the probability of default approaches one as the time goes to infinity. This is expected as the likelihood of default is certain given enough time.

The market participants, in general, will have different expectations for the hazard rate h at different times. The simplest way to obtain a probability of default with a time-dependent hazard rate is to assume a piecewise-flat hazard rate. Other approaches, such as piecewise-linear hazard rates, have been also proposed, see e.g. [1], however the use of piecewise-flat hazard rates has become a market standard and should be preferred.

For example, given a number n of time nodes $t_1, \ldots, t_n$ and n constants $h_1, \ldots, h_n$, we can define $h(t)$ as

$$h(t) \,=\, h_i \qquad \text{if} \qquad t_{i-1} < t \leq t_i \qquad \text{for} \qquad i = 1, \ldots, n\,, \tag{2}$$

and $h(0) = 0$.

An example of graph for $h(t)$ just defined is shown in figure 1. The probability of default can then be computed as

$$P(\tau_{\text{default}} \leq t) \,=\, P(t) \,=\, 1 - \exp\left(-\int_0^t h(\tau) d\tau\right) \,. \tag{3}$$

In particular the probability of default at node i is given by the following recursive relation

$$P_i \,:=\, P(t_i) \,=\, 1 - (1 - P_{i-1}) \exp\left[-h_i (t_i - t_{i-1})\right] \quad \text{for} \quad i = 1, \ldots, n\,, \tag{4}$$

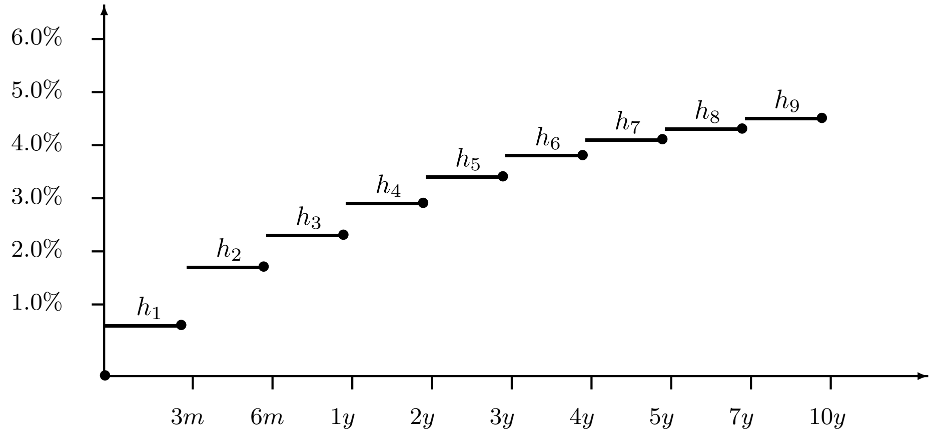

Figure 1: Plot of the piecewise-flat hazard rate $h(t)$ for an hypothetical hazard-rate term structure. Values of the hazard rate pieces h_i, for $i = 1, \ldots, n$, are plotted as ordinates, the nodes from three months (labeled $3m$) to ten years (labeled $10y$) are plotted as abscissa (not on scale).

Where, obviously,

$$P_0 \;=\; P(t_0) \;=\; 0\,. \tag{5}$$

It will be shown in subsection 4 how to obtain a probability term structure, i.e. the values of h_i's, that are consistent with market expectations of defaults.

The conditional probability of default from two different times is another quantity that will be useful in the evaluation of credit derivatives. More precisely, the probability of default as standing at time t, between two future times t_1 and t_2 is given by,

$$P(t_1 < \tau_{\text{default}} \le t_2) \;=\; P(t_1, t_2) \;= P(t_2) - P(t_1)\,. \tag{6}$$

Now that we are familiar with the term structure of default probability, we can continue in the next section to introduce the simplest case of credit derivative: the credit default swap.

3 Credit default swaps

A credit default swap, CDS for short, is a contract between two parties, the buyer and the seller of protection. Credit default swaps have been used for a long time in the credit markets, however, it is only after ISDA defined a standard notion of default in 1999 (see later in this section for more details) that the market has had an explosion in the number of contracts traded.

Further standardization has come in the wake of the credit crunch of 2008. Since March 2009 credit default swaps are often quoted using an up-front payment and a running spread (often 100 basis points for investment grade and 400 for high yield).

Given a certain notional N, the buyer of protection agrees to pay an up-front percentage of N and a fixed rate, known as the CDS spread, on a certain schedule until maturity or the default of the underlying issuer arises. If the default occurs in between coupon payments the buyer of protection agrees also to pay the accrued coupon until default time. In exchange, the seller of protection agrees to buy at face value a bond issued by the underlying name for the same amount as the CDS notional.

Very often an underlying bond, issued by the target name, is used to determine the default event. In case of default the bond value at default is known as the *recovery value* and its ratio to the notional, the recovery ratio, is denoted by R. The difference between the bond face value and the bond market value multiplied by the notional is usually termed *loss given default* and is equal to $(1-R)N$. The exact value of the loss given default might not be known until few weeks after the actual default occurred.

3.1 Computation of net present value for credit default swaps

The NPV computation of a credit default swap paying an upfront percentage u and a spread s on a notional N, can be split into four terms: the upfront payment, a coupon payment leg $s\,NA$, an accrual leg, $s\,NB$, and a default leg NC. Hence, the net present value of the CDS is given by,

$$\mathrm{NPV} \,=\, N\left(u\,U + s\,A + s\,B - C\right)\,, \tag{7}$$

where $U = D(t_s)$ is the discount factor computed at the CDS settlement date t_s (date at which the upfront fee is paid), and A, B, and C are yet to be determined.

Given the up-front percentage u, the break even, or fair spread, is the spread $\tilde{s}$ such that the NPV, defined in equation (7), is zero, i.e.,

$$\tilde{s} \,=\, \frac{C - u\,U}{A+B}\,. \tag{8}$$

3.2 The mid-point approximation for credit default swaps

To evaluate credit default swaps we will follow the method used by Hull and White (see reference [6]). Suppose t_0 is the start accrual date of the first coupon and $t_1, \ldots, t_n$, are the payment coupon dates[1]. We assume the mid-point approximation: coupons are always paid at coupon dates but defaults can only occur exactly half way through a coupon. It can be shown that the parameters A, B, and C, in this case, can be computed as,

$$A = \sum_{i=1}^{n} \left[1 - P(t_i)\right] D(t_i) Y\left(t_{i-1}, t_i\right)\,, \tag{9}$$

$$B = \sum_{i=1}^{n} P(t_{i-1}, t_i) D(t_{i-1/2}) Y\left(t_{i-1}, t_{i-1/2}\right)\,, \tag{10}$$

$$C = \sum_{i=1}^{n} (1-R)\, P(t_{i-1}, t_i) D(t_{i-1/2})\,, \tag{11}$$

with $t_{i-1/2}$ denoting the half point between t_{i-1} and t_i, $D(t)$ is the risk-free discount factor at time t, and $Y(t,s)$ is the year fraction between times t and s using the appropriate day-count convention.

Note that, in the special case where both the instantaneous forward rates and the hazard rate are piecewise flat, the price of a credit-default swap has an analytic solution (see [7] for details). However, in most cases of practical interest the mid-point approximation gives accurate results and, being more general, can be used in a much larger number of cases.

3.3 Definition of default event

At the beginning of the current section we defined a credit default swap as a derivative paying upon certain unspecified default event. It should be noticed that the CDS market did not evolve until

[1] The case in which CDS coupons are paid at the beginning of the period is similar and will be omitted since it does not give any more insight into the pricing process.

a unified definition of default event was found. In practice, the three principal credit events for corporate borrowers are bankruptcy, failure to pay a loan, and the restructuring of a debt. Clearly, the restructuring event is the most likely contingency for a CDS contract. At the time of writing there are at least four different definitions of restructuring. We briefly describe the different type of restructuring and refer to [9] for more details.

Full Restructuring (FR). This 1999 ISDA definition dictates that any restructuring event qualifies as a credit event. Therefore, even a restructuring that increases the value of present and future coupons can trigger the default event.

Modified Restructuring (MR). This 2001 ISDA definition states that a credit event is triggered by all the restructuring agreements that do not cause a loss. While restructuring agreements still trigger a credit event, this clause limits the deliverable obligations to those with a maturity of 30 months or less after the termination date of the CDS.

Modified Modified Restructuring (MM). In 2003 ISDA introduced a new definition of restructuring, similar to Modified Restructuring, where agreements still trigger credit events, but the remaining maturity of deliverable assets must be shorter than 60 months for restructured obligations and 30 months for all other obligations.

No Restructuring (NR). Under this contract condition all restructuring events are excluded as trigger events. Some of the most popular CDS indexes in North America, including those belonging to the CDX index, are traded under this definition.

4 Bootstrapping the probability term structure

Section 2 described how it is possible to price a CDS with the help of a probability term structure $P(t)$ and a risk-free discount factor $D(t)$. In general, one does not want to guess the values of the parameters h_i's but to determine them consistently with the market prices of liquid credit default swaps. In other words we want the probability of default to be risk neutral and to match market expectations.

Therefore, consider a number of CDS contracts referred to a certain issuer for a given restructuring type and seniority. For example, credit-default-swap spreads of the Italian company FIAT observed on May 2006 and July 2007 are reported in table 1. These refer to senior, modified modified restructuring, in currency EUR. Bootstrapping the probability term structure means finding a number of time nodes t_i's and hazard rates h_i's so that the credit default swaps quoted by the market are exactly priced. In order to achieve this goal we set $t_0 = 0$ to the current time, t_1 to the maturity of the first quoted CDS, and h_1 to an arbitrary value to be determined later. The expression for the probability of default $P(t)$ becomes

$$P(t) \,=\, 1 - e^{-h_1\, t} \qquad \text{for} \quad t_0 \le t \le t_1\,. \tag{12}$$

Note that, for the time being, $P(t)$ is not defined for $t > t_1$, however, it is possible to compute $P(t_{1/2})$ and $P(t_1)$,

$$P(t_{1/2}) \,=\, 1 - e^{-h_1\,\frac{t_1}{2}} \quad \text{and} \quad P(t_1) \,=\, 1 - e^{-h_1\, t_1}\,. \tag{13}$$

Using these values, the discount term structure $D(t)$, formulas (9-7) for the NPV, and the quoted spread s_1, it is possible to compute the NPV of the first CDS. It is then easy to find the only h_1

Maturity	May 2006	July 2007
$3m$	22.4	19.6
$6m$	32.0	28.0
$1y$	40.0	35.0
$2y$	79.0	50.0
$3y$	118.0	65.0
$4y$	156.5	82.5
$5y$	195.0	100.0
$6y$	217.5	112.0
$7y$	240.0	124.0
$8y$	251.7	133.7
$9y$	263.3	143.3
$10y$	275.0	153.0
$11y$	275.2	153.2
$12y$	275.4	153.4
$15y$	276.0	154.0
$20y$	277.0	155.0
$30y$	278.0	156.0

Table 1: Example of quoted fair spreads for Senior MM Fiat CDS at two different dates. Spreads are quoted in basis points per year for a number of different maturities. As shown in the text, from these quotes it is possible to bootstrap a term structure for the default probability.

that gives the quoted upfront NPV for the first CDS and therefore match the first quoted CDS spread.

In order to find a value for h_2, we set t_2 to the expiration date of the second CDS and extend the probability structure $P(t)$ so that

$$P(t) \;=\; 1 - [1 - P\,(t_1)]\, e^{-h_2(t-t_1)} \quad \text{for} \quad t_1 < t \leq t_2\,. \tag{14}$$

Notice how $P(t_{1+1/2})$ and $P(t_2)$ are well defined for each value of h_2 and that $P(t)$ is still not defined for $t > t_2$. Using expressions (12) and (14) for $P(t)$, $D(t)$, and (9-7), it is possible to solve for h_2 so that the second CDS is priced to have the quoted NPV, thus matching the second CDS spread as well.

Similarly, given $h_1, h_2, \ldots, h_{i-1}$ the probability term structure is known up to t_{i-1} included. To evaluate h_i set t_i to the maturity of the i-th CDS and define

$$P(t) \;=\; 1 - [1 - P\,(t_{i-1})]\, e^{-h_i(\,t-t_{i-1})} \quad \text{for} \quad t_{i-1} < t \leq t_i\,. \tag{15}$$

Finally, solve for h_i in order to obtain the quoted NPV for the i-th CDS. This procedure can be continued until the last CDS is priced, obtaining an expression for the probability structure up to the maturity of the last CDS so that all the CDS spreads are matched.

5 Bonds, asset swaps, and z-spreads

The knowledge of the default-probability term structure is used in pricing financial products while taking into account the credit worthiness of their issuers. In this section we show how to price a fixed-rate coupon bond in presence of credit risk, we introduce the par asset swap, and define the zero-volatility spread.

5.1 Pricing a bond in presence of credit risk

Consider the problem of pricing a fixed-rate coupon bond issued by a risky entity like a corporate bond. Given one or more CDS quotes on the issuer, the recovery ratio R, and a risk-free term structure for the discount $D_r(t)$, it is possible to build a term structure for the probability of default $P(t)$ as shown in section 3. The present value of each cash flow paid at time t must then be multiplied by the survival probability at that date and then discounted on on the risk-free term structure $D_r(t)$. It is assumed that in case of default the full recovery value is paid back. Therefore the net present value of a fixed-rate coupon bond is given by

$$\begin{aligned} \mathrm{NPV}_{\mathrm{bond}} = \sum_{i=1}^{n} \left[1 - P(t_i)\right] D_r(t_i)\, N\, C_i + \left[1 - P(t_n)\right] D_r(t_n)\, N + \\ + N\, R \int_0^{t_n} D_r(\tau)\, P(\tau + d\tau|\tau)\,, \end{aligned} \tag{16}$$

where N is the bond notional, t_i, for $i = 1, \ldots, n$, are the dates of future coupon payments, $N\, C_i$ is the value of the coupons to be paid, and, finally, $P(\tau + d\tau|\tau)$ is the probability of default between time τ and $\tau + d\tau$ conditional to no default at time τ (note that this term is proportional to $d\tau$). The first line represents the part of expected cash flows in case of survival, while the second line the cash flows in case of default, i.e. $R\, N$, to be paid at the random time of default τ. The dirty price of the bond can be computed from its NPV

$$P_{\mathrm{bond}}^{\mathrm{dirty}} = \frac{100}{N} \frac{\mathrm{NPV}_{\mathrm{bond}}}{D_r(t_b)}\,, \tag{17}$$

where t_b is the bond settlement date and no defaults are assumed between the current date and the bond settlement date.

5.2 The par asset swap

A par asset swap, or more loosely an asset swap, is a product that allows the holder of a fixed-coupon bond from a certain issuer, to swap the fixed cash flow string obtained from the bond to a stream of cash flows based on the Libor rate plus a spread. In swapping the cash flows the investor retains the original credit worthiness of the issuer. Unlike a standard interest-rate swap, the premium is based on the credit exposure to the bond issuer and not on that of the asset-swap issuer.

The asset swap is composed of a fixed leg that pays exactly the same coupons as the original bond, but not the redemption, and a floating leg that pays the Libor rate plus the asset swap spread A. The floating payment is usually made at a different frequency and accrual basis than the fixed payment. The net present value of the asset swap at the bond settlement date, can be computed as

$$NPV_{\mathrm{asw}}(t_b) = N \left(L_{\mathrm{libor}} + A\, L_{\mathrm{spread}} - L_{\mathrm{fixed}}\right)\,, \tag{18}$$

with

$$L_{\mathrm{libor}} = \sum_{i=1}^{n_{\mathrm{float}}} D_r(t_b, \tau_i)\, L_{i-1} Y\left(\tau_{i-1}, \tau_i\right) - L_0 Y\left(\tau_0, t_b\right)\,, \tag{19}$$

$$L_{\mathrm{spread}} = \sum_{i=1}^{n_{\mathrm{float}}} D_r(t_b, \tau_i)\, Y\left(\tau_{i-1}, \tau_i\right) - Y\left(\tau_0, t_b\right)\,, \tag{20}$$

$$L_{\mathrm{fixed}} = \sum_{i=1}^{n_{\mathrm{fixed}}} D_r(t_b, t_i)\, C_i\,, \tag{21}$$

where, for the fixed cash flows, we retain the same conventions as those in section 5.1, τ_i are the payment dates for future floating payments with $i = 1, \ldots, n_{float}$, τ_0 is the accrual date for the current floating coupon, and $Y(\tau_1, \tau_2)$ is the year fraction between τ_1 and τ_2 for the floating leg; finally, for each t, $D_r(t_b, t)$ is the *forward* discount factor between t_b and t given by,

$$D_r(t_b, t) = \frac{D_r(t)}{D_r(t_b)} . \tag{22}$$

Note that the asset swap can be bought at any time during the life of the underlying bond. When it is bought in between coupon dates the next fixed coupon of the asset swap will completely match that of the bond and no accrual is therefore paid. Given the bond price $P^{\rm dirty}_{\rm bond}$, obtained either from the market or from equation (16), the asset swap is said to be at par when the portfolio composed by the bond and the asset swap are equivalent to a bond at par, i.e.,

$$P^{\rm dirty}_{\rm bond} + 100 \, \frac{{\rm NPV}_{\rm asw}(t_b)}{N} = 100 . \tag{23}$$

The asset swap spread A can be easily computed from equations (18) to (23) as

$$A = \frac{1 - P^{\rm dirty}_{\rm bond}/100 + L_{\rm fixed} - L_{\rm libor}}{L_{\rm spread}} . \tag{24}$$

Similarly, given the asset swap spread for a certain bond, it is possible to compute the implied bond price:

$$\frac{P^{\rm dirty}_{\rm bond}}{100} = 1 + L_{\rm fixed} - L_{\rm libor} - A\, L_{\rm spread} . \tag{25}$$

Therefore, the asset-swap spread and the bond price are equivalent and the knowledge of one is necessary and sufficient to determine the other.

5.3 Zero Volatility Spread

The zero volatility spread, or z-spread, is the continuously compounded spread Z that should be applied to the Libor curve in order to price a bond consistently with market quotes. Given a fixed-rate coupon bond, such as that described in section 5.1, its z-spread Z is such that

$${\rm NPV}_{\rm bond} = \sum_{i=1}^{n} e^{-Z\, t_i} D_{\rm Libor}(t_i)\, N\, C_i + e^{-Z\, t_n} D_{\rm Libor}(t_n)\, N , \tag{26}$$

where $D_{\rm Libor}(t)$ is the discount at time t on the Libor curve. Notice that the z-spread is equivalent to the bond price and to the asset-swap spread for the matter of pricing. The z-spread is used, for example, in computing the simulation of credit risk as explained in reference [3].

6 Portfolio credit derivatives and copula models

So far we have examined credit derivatives whose payoff is based on the default of a single name. In practice there is large class of credit derivatives whose payoff depends on the default of more than one name: the so-called portfolio credit derivatives. The remainder of this paper describes portfolio credit derivatives and how they can be priced using a copula model.

As described in section 3, a credit default swap allows the purchase, or the sale, of default insurance on a single reference company. However, in practice a financial institution is usually exposed to the credit risk of a portfolio of names. Furthermore, a certain amount of loss in the given portfolio can be borne, hence there is demand for protection on the credit loss coming from a number of names. In order to satisfy this demand the market offered a number of products, known as portfolio credit derivatives, the most common of which are the collateralized debt obligation.

For single-name credit derivatives the probability of default is the key ingredient to pricing. As shown in the next section, in portfolio credit derivatives this role is taken by the portfolio loss distribution: the distribution of the accumulated portfolio losses due to defaulting underlying names. In this section we analyze the particular case of a loss distribution obtained by the standard copula model.

6.1 General formulation of copula models

A copula is a method to specify the joint probability distribution of a number of random variables so that the marginal distributions on a single variable matches the original distribution. We leave the description of the generic framework of copulas to specialized texts, see e.g. reference [2], instead we show a simple formulation that can be easily applied to the generation of the loss distribution for a credit portfolio.

We will consider a general formulation that covers a fairly large number of copula models and then specialize to the popular case of Gaussian copula.

Mapping of default events Given a credit instrument based on a portfolio of n names, each with a given cumulative default distribution $P_i(t)$, we show how to define a sensible joint distribution for all the names so that its marginal distributions on any given name matches the single name distribution. Consider a fixed time horizon T, assume that the default event of name i, with $i = 1, \ldots, n$, happens if and only if a certain random variable y_i, to be defined more precisely later, drops below a certain level ξ_i,

$$\text{name } i \text{ defaults before time } T \quad \Longleftrightarrow \quad y_i < \xi_i \,. \tag{27}$$

A cumulative probability function F, to be specified later, is used to map each ξ_i to T,

$$\xi_i = F^{-1}\left(P_i(T)\right) \,. \tag{28}$$

Even though we will assume this definition as axiomatic, it is possible to derive equation (28) in many different types of structural models (see, for example, the model of Vasicek in reference [11]).

Once the random variables $y_1, \ldots, y_n$ have been defined, and we will do it shortly, it is necessary to make sure that their marginal distribution is the same as that of the single names $P_i(t)$. In order to achieve this result, the random y_i's are mapped percentile-to-percentile onto the corresponding default times t_i,

$$t_i \longleftrightarrow y_i \,, \tag{29}$$

using the y_i's cumulative distribution G_i, i.e.,

$$\bar{y}_i = G_i^{-1}\left(P_i(\bar{t}_i)\right) \quad \text{for } i = 1, \ldots, n \,. \tag{30}$$

Since both F and G are monotone increasing functions, it is possible to express the random times $\bar{t}_i$'s in terms of the variables y_i's,

$$\bar{t}_i = P_i^{-1}(G_i(\bar{y}_i)) \quad \text{for } i = 1, \ldots, n \,. \tag{31}$$

A consequence of this mapping is the following map on the events,

$$\text{Event}(t_i < T) \quad \text{is equivalent to} \quad \text{Event}(y_i < \xi_i) \,, \tag{32}$$

where the term *equivalent* means that one event happens if and only if the other event happens.

Definition of the random variables y_i's Given an integer M, define n random variables y_i, one for each name, so that,

$$y_i \;=\; a_{i,1}\,W_1 + a_{i,2}\,W_2 + \ldots + a_{i,M}\,W_M + Z_i\sqrt{1 - a_{i,1}^2 - a_{i,2}^2 - \ldots - a_{i,M}^2}\,, \tag{33}$$

where the W_j (with $j = 1, \ldots, M$) and the Z_i (with $i = 1, \ldots, n$) are independent, identically distributed[2], random variables with zero mean and unit variance. Furthermore, we assume G to be the cumulative distribution of each W_j and Z_i, and, in the Gaussian-copula model, G is the normal cumulative distribution.

Equation (33) describes an M-factor copula model, however, in the present paper, we will restrict ourselves to the popular one-factor models, in which equation (33) is written with $M = 1$,

$$y_i = a_i W + \sqrt{1 - a_i^2} Z_i. \tag{34}$$

The dependence of random variables y_i's on the common factor W makes them all correlated. The correlation between the variables y_i and y_j can be computed as

$$\begin{aligned}
\langle y_i y_j \rangle &= \left\langle \left(a_i W + \sqrt{1 - a_i^2} Z_i \right) \left(a_j W + \sqrt{1 - a_j^2} Z_j \right) \right\rangle = \\
&= \left\langle a_i a_j W^2 + \sqrt{1 - a_i^2}\sqrt{1 - a_j^2} Z_i Z_j + a_i \sqrt{1 - a_j^2} Z_j W + a_j \sqrt{1 - a_i^2} Z_i W \right\rangle = \\
&= a_i a_j + \sqrt{1 - a_i^2}\sqrt{1 - a_j^2}\,\langle Z_i Z_j \rangle + 0 + 0,
\end{aligned}$$

since $\langle W Z_j \rangle = 0$ and $\langle Z_i Z_j \rangle$ is 1 if $i = j$ and zero otherwise. In summary,

$$\langle y_i y_j \rangle = \begin{cases} 1 & \text{when } i = j \\ a_i a_j & \text{when } i \neq j \end{cases}\,. \tag{35}$$

As a consequence of the event-to-event mapping, see equation (32), the conditional probabilities for the y_i are equal to those for the default times t_i's and, therefore, given a value for W,

$$\mathrm{Prob}(t_i < T|W) = \mathrm{Prob}(y_i < \xi_i|W)\,. \tag{36}$$

Since W is fixed, using equation (34) and some algebraic manipulation, the probability of default conditional to W becomes,

$$\begin{aligned}
\mathrm{Prob}(y_i < \xi|W) &= \mathrm{Prob}\left(a_i W + \sqrt{1 - a_i^2} Z_i < \xi|W \right) = \\
&= \mathrm{Prob}\left(Z_i < \frac{\xi - a_i W}{\sqrt{1 - a_i^2}}|W \right) = G\left(\frac{\xi - a_i W}{\sqrt{1 - a_i^2}} \right)\,.
\end{aligned} \tag{37}$$

This expression will be used later in the computation of default probability of multiple defaults.

Monte Carlo simulation of default times In the framework that we have just developed it is easy to simulate the default times up to time T. Proceed as follows:

1. Define the cross correlations from the numbers $a_1, \ldots, a_n$ and equation (35)

[2]This assumption is not strictly necessary, however, it will be used since it makes the notation simpler.

2. Draw $n+1$ independent random numbers $W, Z_1, \ldots, Z_n$ distributed as G, for example Gaussian, with mean 0 and variance 1

3. Using equation (34) compute the values of $\bar{y}_1, \ldots, \bar{y}_1$

4. Determine the simulated default times $\bar{t}_1, \ldots, \bar{t}_n$, from expression (31)

The random default times $\bar{t}_1, \ldots, \bar{t}_n$, thus obtained are correlated, see equation (35), and each of them is distributed with probability $P_i(t)$ by construction.

Even though a Monte Carlo simulation is always feasible, we are going to increase significantly the computational efficiency by integrating semi-analytically some of the average quantities obtained by the random variables $y_1, \ldots, y_n$.

We have just described the essence of a copula model. Expression (37) for the conditional probability of default has been obtained in general and is not necessarily related to a particular choice for the probability distribution of the random numbers y_i's. In the following subsection we specialize this general formulation to the case of a Gaussian copula.

6.2 Special case of a Gaussian copula model

Formula (37) for the conditional cumulative distribution of y_i's can be readily transformed in that of the random times t_i's using equation (36):

$$\mathrm{Prob}(t_i < T|W) = G\left(\frac{F^{-1}(P_i(T)) - a_i W}{\sqrt{1-a_i^2}}\right). \tag{38}$$

It is a standard market practice to consider the same value $\sqrt{\rho}$ for all parameters $a_1, \ldots, a_n$, and to assume a Gaussian cumulative distribution for both F and G. In this case equation (38) simplifies to

$$\mathrm{Prob}(t_i < T|W) = \Phi\left(\frac{\Phi^{-1}(P_i(T)) - W\sqrt{\rho}}{\sqrt{1-\rho}}\right), \tag{39}$$

where Φ is the standard cumulative normal distribution. Notice that the conditional default probability $\mathrm{Prob}(t_i < T|W)$ depends solely on the parameter ρ that can be interpreted, in virtue of equation (35), as the correlation of default times.

There are two different cases in which equations (38) and (39) can be applied: the case in which all the names are very similar, denoted as the homogeneous case, and the case in which the underlying names are not particularly similar: the non-homogeneous case. The remainder of this section illustrates how to use these expressions for the conditional probability of default to compute the expected loss distribution.

6.3 Homogeneous case

The homogeneous case is the case in which the credit losses are proportional to the number of defaulted names and not to the particular name for which the default happened. Many of the popular quoted single tranche CDOs fall into this category. For example, in a first-to-default basket it is usually assumed that the expected recovery ratio is the same for all the names. In other words, we are in the homogeneous case if and only if the default losses are proportional to the number of defaulted assets.

Thus, the (conditional) survival probability of the i-th name up to time T is given by

$$S_i(t|W) \;=\; 1 - G_i\left(\frac{G_i^{-1}(P_i(t)) - a_i W}{\sqrt{1-a_i^2}}\right). \tag{40}$$

Since the Z_i's are independent, it follows that, conditionally on W, the default times t_i's are also independent. Therefore, the probability that no defaults occur before time t is simply given by the joint probability that all the names survive,

$$\pi_0(t|W) = \prod_{i=1}^{n} S_i(t|W). \tag{41}$$

In a similar way, the probability of exactly one default before time T is given by the probability that any name defaults while all the others survive, i.e.,

$$\pi_1(t|W) \;=\; \sum_{i=1}^{n} \left\{ [1 - S_i(t|W)] \prod_{j\neq i} S_j(t|W) \right\}. \tag{42}$$

In general, the probability of exactly k defaults before t is given by

$$\pi_k(t|W) = \sum_{\mathcal{D}} \left\{ \prod_{i\in\mathcal{D}} [1 - S_i(t|W)] \prod_{i\notin\mathcal{D}} S_i(t|W) \right\}, \tag{43}$$

where $\mathcal{D}$ is a set of k distinct numbers chosen from $\{1, 2, \ldots n\}$ and the summation is taken over the $\frac{n!}{k!(n-k)!}$ ways in which $\mathcal{D}$ can be chosen.

Equation (43) can be easily evaluated for each k by means of polynomial algebra as shown in reference [7]. For each name, let us define the binomial $q_i + p_i\xi$, where ξ is a dummy variable, $q_i = S_i(t|W)$ is the (conditional) survival probability and $p_i = 1 - q_i$ is the default probability. Let us now consider the polynomial

$$\mathcal{P}(\xi) = \prod_i (q_i + p_i\xi). \tag{44}$$

It is easy to deduce that the order-zero term in $\mathcal{P}(\xi)$ is the product of all the q_i, i.e., the probability that no defaults occur before t. The first-order term is a sum of n monomials, their coefficients being each the product of exactly one p_i and all the q_j for $j \neq i$. As seen above, such sum corresponds to the probability of exactly one default. In general, the order-k term as the sum of $\frac{n!}{k!(n-k)!}$ monomials, each one being the product of k terms of the form $p_i x$ and $n - k$ terms of the form q_i. Moreover, such monomials cover all the possible ways in which such a product can be composed. Therefore, we obtain the result that the k-th order coefficient of $\mathcal{P}(x)$ equals the conditional probability $\pi_k(t|W)$ of exactly k defaults by time t.

In the homogeneous case, all names have the same recovery rate R and the same notional N, hence each default results in the same loss $L^R = N\,(1 - R)$ and $\pi_k(t|w)$ maps directly into the loss distribution density $l(x; t, W)$ as

$$l(x; t, W) = \sum_{k=0}^{n} \pi_k(t|W) \cdot \delta(x - kL^R). \tag{45}$$

Finally, the unconditional probability $\pi_k(t)$ of exactly k defaults before time t, can be obtained by integrating $\pi_k(t|W)$ over the distribution of W. This can be done numerically in an efficient way using Gaussian quadrature. The unconditional loss distribution becomes

$$l(x; t) = \sum_{k=0}^{n} \pi_k(t) \cdot \delta(x - kL^R)\,, \tag{46}$$

where,

$$\pi_k(t) = \int_{-\infty}^{+\infty} \pi_k(t|W)\, dF(W)\,. \tag{47}$$

It will be shown later in the paper how this distribution can be used to compute a fair value for single-tranche CDOs.

6.4 Non-homogeneous case

When each name can contribute to a different loss in case of default, equation (45) no longer holds and a continuous distribution is needed. Different approaches can be used to obtain $l(x)$ in this case; the one we will describe here, called *bucketing*, is due to Hull and White and is fully described in reference [6].

For any given t and W, as in the previous subsection, let us define $q_i = S_i(t|W)$ as the (conditional) survival probability of the i-th name and $p_i = 1 - q_i$ as its default probability. We can compute the conditional loss distribution $l(x; t, W)$ with an iterative process as follow: let us divide the set of all possible total losses into N_b buckets $[x_j, x_{j+1}]$ with $x_0 = 0$ and x_{N_b} being the maximum loss attainable. For each bucket k, we will store two quantities, namely, the probability P_k that the total loss is in bucket k, and the loss amount L_k conditional to the total loss being in bucket k.

Now, let us iteratively add names to the portfolio and compute the loss distribution. With no names, obviously, the total loss is null, therefore $P_0 = 1$, $L_0 = 0$, and $P_k = 0$ for all other k (for definiteness, L_k can be set to any value in the bucket so let us assume a zero value.) At this point, consider the addition of a new name i, we can create another set of initially empty buckets with the same x_j's. Such name has a survival probability q_i and a default probability p_i with a corresponding loss l_i. For each original bucket with $P_k \neq 0$, we take its expected loss L_k. With probability q_i, the name will survive; therefore, we add $P_k \cdot q_i$ to the (new) bucket containing L_k. With probability p_i, the name will default; therefore, we add $P_k \cdot p_i$ to the bucket containing $L_k + l_i$. If the target bucket is not empty, we will add the new probability to the existing one and set the expected loss to the weighed average of the existing one and the one being added.

When all names are added, the resulting set of buckets gives us the conditional loss distribution $l(x; t, W)$ as

$$l(x; t, W) = \sum_{k=0}^{N_b} P_k(t, W) \cdot \delta(x - L_k). \tag{48}$$

Again, as in the homogeneous case, this expression will be used to compute a fair price for single-tranches CDOs.

7 Pricing portfolio credit derivatives

7.1 Basic instrument definitions

The most popular portfolio credit derivative is known as credit debt obligation, or CDO for short. There are many different types of CDOs, however, we will consider the commonly quoted single-tranche CDO on a basket of credit default swaps. This instrument can be thought as insurance against the losses incurred in the given portfolio of credit default swaps from a certain percentage of the portfolio notional (the attachment point), to another percentage (the detachment point). For example, consider a single-tranche CDO on the CDX North America Investment Grade basket, CDX.NA.IG for short, with an attachment point of 4% and a detachment at 7%. Table 2 shows the attachment/detachment points for the most popular quoted single-tranche CDOs. Suppose that at

iTraxx Europe	0–3%	3–6%	6–9%	9–12%	12–22%
CDX North Am. Inv. Grade	0–4%	4–7%	7–10%	10–15%	15–30%
CDX North Am. High Yield	0–10%	10–15%	15–25%	25–35%	

Table 2: Attachment–Detachment points for some popular traded single-tranches CDOs

the beginning of the deal there are 125 names and for each name there is an underlying CDS with a notional of 100,000 \$. Also assume that the deal is made for five years, the beginning of the deal is at time t_0 and the protection buyer pays quarterly fees at times $t_1, \ldots, t_n$, corresponding to an annual rate of $s = 62.0$ basis points. The amount under protection is therefore given by,

$$N^t \;=\; (7\% - 4\%) \times 125 \times 100,000\,\$ \;=\; 375,000\,\$ \tag{49}$$

and this is precisely the notional over which the spread s is to be paid. The protection seller does not pay anything until the portfolio incurs into a loss equal to the attachment point multiplied by the portfolio outstanding, that is the attachment notional amount N^a,

$$N^a \;=\; 4\% \times 125 \times 100,000\,\$ \;=\; 500,000\,\$\,. \tag{50}$$

Suppose that some years after the deal, there are unfortunate events such that the credit losses of the portfolio reach exactly the value N^a. At this point the *insurance* from the protection seller starts to pay out. Suppose further that after few more months, at a certain time $\bar{t}$, there is another default that results in a further portfolio loss of 75,000 \$. This loss triggers the following events:

- a cash value of 75,000 \$ is paid by the protection seller to the protection buyer
- the notional of the single-tranche CDO is reduced by 75,000 \$ to 300,000 \$

All the subsequent payments will be based on this new notional tranche amount until further losses occur.

Finally, suppose that after some more time there are exactly three defaults with zero recovery, for a total loss of 300,000 \$. After these events the protection seller would have paid a total of 300,000 \$ to the protection buyer, hence the tranche notional is reduced to zero, and the contract ceases to exist.

7.2 Pricing CDOs

The main factor affecting the price of portfolio credit derivatives is without a doubt the combined probability of default of the underlying names. Here we give an expression of the expected tranche loss using the copula model described in section 6.

Given the conditional loss distribution $l(x; t, W)$, the unconditional loss distribution can be computed as,

$$l(t; x) = \int_{-\infty}^{+\infty} l(x; t, W)\, d\phi(W) = \int_{-\infty}^{+\infty} l(x; t, W)\, \phi(W)\, dW, \tag{51}$$

where ϕ is the standard Gaussian density. One can then define, for example, the average loss at time t as

$$L(t) = \int_0^{L_M} l(x; t)\, dx \tag{52}$$

where L^M is the maximum loss possible, so that for any given t, i.e., we have $l(x; t) = 0$ for all $x > L^M$. Also, given a financial variable G depending on the portfolio loss x, and, possibly, on time, its expected value at time t can be computed as,

$$\mathbb{E}_t\left[G \circ L\right] = \int_0^{L_M} G(x;t) l(x;t)\, dx\,, \tag{53}$$

where the symbol '$\circ$' reminds us of function composition. $\mathbb{E}_t[G \circ L]$ could also be written $\mathbb{E}_t[G(L)]$, so that definition (53) can be seen as the average value of G on the stochastic loss L.

An insight on the meaning of $\mathbb{E}_t[G \circ L]$can be inferred by computing the expected value of the identity function I, i.e. the function that maps x to itself,

$$L(t) = \mathbb{E}_t\left[I \circ L\right]\;; \tag{54}$$

therefore, $\mathbb{E}_t\left[G \circ L\right]$ has the meaning of the expected value of a function of the loss. Using this notation it is easier to write the contribution to the present value of all the cash flows that form a CDO.

Cash flow at a known time Consider first the payment of the fixed coupon: a cash-flow that pays at a fixed time t_i an amount $C(x)$ depending on the realized loss x. For example, consider the fixed payment for a tranche that pays from a loss L_a at the attachment point to a loss L_d at the detachment point. Such a cash flow can be written as

$$C(x) = N^t\, s\, Y(t_{i-1}, t_i)\,\left[1 - H(x)\right]\,, \tag{55}$$

where N^t is the remaining tranche notional, s is the percentage spread to be paid, $Y(t_{i-1}, t_i)$ is the accrual time computed on the relevant day-counter convention, and $H(x)$ is a function of the loss and is given by

$$H(x) = \begin{cases} 0 & x < L_a \\ \frac{x-L_a}{L_d-L_a} & L_a \le x \le L_d \\ 1 & x > L_d \end{cases}\;. \tag{56}$$

The expected discounted cash-flow is, therefore,

$$\mathrm{PV} = D(t_i)\, \mathbb{E}_{t_i}\left[C \circ L\right]\,, \tag{57}$$

where $D(t_i)$ is the appropriate discount factor at time t_i.

Default leg cash flows A different type of cash flow occurs for an instrument when one or more underlying name defaults. The challenge is that it is not known the exact time of default t and that the resulting cash flow is proportional to $F(L(t+)) - F(L(t-))$ where, with an abuse of notation, $L(t+)$ is loss right after time t and $L(t-)$ is the loss just before t. In the case of the credit leg of a CDO, F is given by,

$$F(x) = N^t\, H(x)\,, \tag{58}$$

where N^t is the remaining notional amount of the CDO tranche, and $H(x)$ is given by equation (56).

Consider a small interval of time between t and $t + dt$: conditional to a given loss x, the loss at time $t + dt$ can be written as $x + dx$. Hence, the discounted cash flow resulting from the loss is given by

$$D(t)\left[F(x + dx) - F(x)\right]\,, \tag{59}$$

multiplied by the conditional probability (density) of a default between time t and time $t + dt$ which, neglecting higher order terms, is given by

$$l(t + dt; x + dx) - l(t; x)\,. \tag{60}$$

Using the notation of expression (53) the expected discounted value of all such cash-flows up to time T can be written as

$$\mathrm{PV} = \int_0^{t_n} D(t)\, \mathbb{E}_t \left[d(F \circ L) \right] . \tag{61}$$

As noted by Joshi in reference [7], this expression can be integrated by parts to obtain

$$\mathrm{PV} = D(t_n) \mathbb{E}_{t_n} \left[F \circ L \right] - \int_0^{t_n} \frac{dD}{dt}\, \mathbb{E}_t \left[F \circ L \right] dt \,. \tag{62}$$

Alternatively, one can use the mid-point approximation of next subsection to compute the integral in equation (61).

Cash flows of accrual leg In certain traded single-tranche CDOs the amount paid by the fixed leg is proportional to the weighed-average tranche notional during the life of the coupon. Therefore, it is necessary to subtract a small accrual leg to the fixed coupon already computed in equation (57). In other words for any fraction of time for which the coupon is computed, it is necessary to subtract the corresponding payment in the default leg.

For a given coupon starting at time t_i and ending at time t_{i+1} the accrual is proportional to

$$\int_{t_i}^{t_{i+1}} \mathbb{E}_t \left[Y\left(t, t_i\right) B(t) d(F \circ L) \right] . \tag{63}$$

Net present value of CDO Summarizing, the CDO's net present value (NPV) can be written as

$$\mathrm{NPV} \;=\; N^t \left(u\, U + s\, A + s\, B - C \right) \,, \tag{64}$$

for any given upfront percentage u and spread s. The NPV was split into four legs: the upfront payment $u\,U$, the coupon payment leg $s\,N^t A$, the accrual coupons, $s\,N^t B$, and the default leg $N^t C$, where,

$$U \;=\; D(t_s)\,, \tag{65}$$

$$A \;=\; \sum_{i=1}^{n} D(t_i) Y\left(t_{i-1}, t_i\right) \mathbb{E}_{t_i} \left[(I - H) \circ L \right] \,, \tag{66}$$

$$B \;=\; -\sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} Y\left(t, t_i\right) D(t)\, \mathbb{E}_t \left[d(H \circ L) \right] \,, \tag{67}$$

$$C \;=\; \int_0^{t_n} D(t)\, \mathbb{E}_t \left[d(H \circ L) \right] \,, \tag{68}$$

where t_s is the settlement date of the CDO.

For equity tranches, the spread s is usually held fixed to $\bar{s} = 500$ basis points and the NPV is quoted in terms of the percentage upfront payment $\tilde{u}$ that brings the NPV to zero. We therefore have,

$$\tilde{u} \;=\; \frac{\bar{s}\,(A + B) - C}{U} \,. \tag{69}$$

For mezzanine and senior tranches the upfront payment is usually zero and the fair spread $\tilde{s}$ is such that the NPV is zero. From equation (64) it follows that

$$\tilde{s} \;=\; \frac{C}{A + B} \,. \tag{70}$$

Equations (66-68) give a useful abstract interpretation of the different components of the CDO value. However, it is still necessary to make further assumptions in order to compute the coefficients A, B, and C. This is accomplished in the next subsection that provides one way to to evaluate numerically the coefficients A, B, and C.

7.3 The mid-point approximation for CDOs

The mid-point approximation used to evaluate credit default swaps introduced in subsection 3.2 can be also applied to obtain a price for CDOs(see reference [6]). In order to compute the net present value of the CDO for any time t, we define the average loss for the default leg,

$$\mathcal{E}(t) = \mathbb{E}_t \left[H \circ L \right] , \tag{71}$$

with H defined by equation (56).

Consider an unseasoned single-tranche CDO and assume t_0 to be the start accrual date of the first coupon and $t_1, \ldots, t_n$, the coupon payment dates[3]. Assume that coupons are always paid at coupon dates and that defaults, one or more, can only occur exactly half way through the life of a coupon. Using this approximation the net present value of the CDO can be computed using expression (64), with U is given by equation (65), $\mathcal{E}(t)$ defined by (71), and A, B, C approximated by,

$$\begin{aligned}
A &= \sum_{i=1}^{n} \left[1 - \mathcal{E}(t_i)\right] D(t_i) Y\left(t_{i-1}, t_i\right) , && (72) \\
B &= \sum_{i=1}^{n} \left[\mathcal{E}(t_i) - \mathcal{E}(t_{i-1})\right] D(t_{i-1/2}) Y\left(t_{i-1}, t_{i-1/2}\right) , && (73) \\
C &= \sum_{i=1}^{n} (1 - R) \left[\mathcal{E}(t_i) - \mathcal{E}(t_{i-1})\right] D(t_{i-1/2}) , && (74)
\end{aligned}$$

where $t_{i-1/2}$ denotes the half point between t_{i-1} and t_i, $D(t)$ is the risk-free discount factor at time t, and $Y(t_{i-1}, t_i)$ is the year fraction between t_{i-1} and t_i computed using the appropriate day count convention.

Let us compute $\mathcal{E}(t)$ more explicitly, in the case of a homogeneous pool where $l(x, t)$ is given by expression (46):

$$\mathcal{E}(t) = \sum_{k=0}^{n} \pi_k(t) \, H(kL^R) \, . \tag{75}$$

In the non homogeneous case, where $l(x, t)$ is given by expression (48), the expected average loss for the default leg at time t can be computed as,

$$\mathcal{E}(t) = \sum_{b=0}^{N_b} P_b(t) H(L_b) \, , \tag{76}$$

where, for each t, the coefficients $P_b(t)$ are given by

$$P_b(t) = \int_{-\infty}^{+\infty} P_b(T, W) \, \phi(W) \, dW; \tag{77}$$

that can be computed numerically using Gaussian quadratures.

Notice how equations (72-74) only need the evaluation of $\mathcal{E}(t)$ at the time nodes $t_1, \ldots, t_n$; this is the main reason why this approach has become so popular among practitioners.

[3] Again, similarly to credit default swaps, the case in which the tranche coupons are paid at the beginning of the period is similar and will be omitted since it does not give any further (insight to the pricing process.

7.4 The standard market model: base correlation

In standard market practice CDO tranches are quoted by means of a single correlation figure, namely, all pairwise correlations in equation (35) are set to a single correlation ρ, leading to all a_i being equal to $\sqrt{\rho}$. In this model, it is, in principle, possible to imply such correlations from the prices of the tranches. However, this can lead to difficulties in case of mezzanine tranches since their price is not a monotone function of correlation hence implying the existence of prices for which the implied correlation is not uniquely defined.

It has become a market standard to write any given tranche as the difference of two equity tranches. The present value of a tranche with attachment point a and detachment point d is written as

$$\mathrm{PV}(a,d) = \mathrm{PV}(0,d) - \mathrm{PV}(0,a)\,. \tag{78}$$

The implied correlation of equity tranches with different detachment points, known as *base correlation*, is quoted in the market and can be used together with equation (78) to price tranches with arbitrary attachment and detachment points (possibly by interpolating between quoted base correlations).

Alternatives to base correlation. Among others Torresetti *et al.* in reference [10] give an alternative formulation to the use of *base correlation*. With the help of expressions (72–74), they show that it is possible to bootstrap directly an expression for $\mathcal{E}(t)$ at nodes $t_1, \ldots, t_n$ as a function of the attachment point.

Although their approach is very interesting as it moves the interpolation/extrapolation problem from base correlation to the more tractable expected loss $\mathcal{E}(t)$, it cannot be applied directly when a separation of the default probability effect and the correlation effect is needed. For example, it cannot be used directly when implementing a credit spread simulation framework such as that developed by Cintioli and Marchioro and described in reference [3]. For this reason, in order to compute historical-simulation scenarios for CDO prices, we will stick to the standard copula approach.

7.5 Historical-simulation scenarios for base correlation

The computation of risk for CDOs and CDO tranches is a relatively new topic in risk management (see reference [4]). It is sometimes necessary to compute the value of CDO tranches on scenarios that are different from current market conditions, possibly in order to compute risk measures or to compute stress scenarios. Consider, for example, the framework of simulated scenarios described by Marchioro and Cintioli in reference [8]. In order to apply such methodology to single tranches CDOs, it is necessary to obtain an expression for the simulated base correlation. The simplest and most straightforward way to obtain these scenarios is the historical simulation of base correlation.

Consider the base correlation of a basket for a certain detachment point d. Given the historical daily fixing ρ_i^{hist}, it is possible to compute, as in reference [8], the historical "returns" of correlation[4] as

$$S_i = \frac{\rho_i^{\mathrm{hist}}}{\rho_{i-1}^{\mathrm{hist}}}\,. \tag{79}$$

The simulated scenarios for base correlation, therefore, can be defined as (see reference [8]),

$$\rho_i = \rho_t\, S_i\,. \tag{80}$$

The scenarios thus obtained should be computed for all the detachment points and all given indexes.

[4]Clearly these returns have nothing to do with the cash return that one would have if she held the CDO tranche in her portfolio.

Notice that, in principle, scenarios obtained using definition (80) may result in a base correlation that is larger than one or in a term structure for the expected losses that is not arbitrage free, however, in practice these cases happen very rarely and can be omitted from the scenarios.

8 Conclusion

In this paper we introduced popular credit derivatives and a summary of the standard market practices used to compute their fair, i.e. arbitrage free, value.

We have shown the importance of a single-name probability term structure for pricing credit default swaps and of the loss distribution of multiple defaults for pricing single-tranche CDOs. While for credit default swaps we believe that the pricing method shown leaves little to be improved, there are several possibilities for further enhancing the pricing of single-tranche CDOs. One of the most promising of these methods is based on the variance-gamma distribution and is described in reference [7].

We hope that our description of credit derivatives, simple but rigorous, will be useful to introduce people of different backgrounds to this subject and to prepare the ground for more sophisticated pricing techniques.

References

[1] Damiano Brigo and Fabio Mercurio. *Interest Rate Models: Theory and Practice.* Springer Finance, Heidelberg, 2^{nd} edition, 2006.

[2] Umberto Cherubini, Elisa Luciano, and Walter Vecchiato. *Copula methods in finance.* Wiley Finance, 2004.

[3] Dario Cintioli and Marco Marchioro. Integrating default risk in the historical simulation model. In *The Simulation Model*, Quantitative Research Series. StatPro website, June 2007. Previews available at `http://marchioro.org`.

[4] Michael S. Gibson. Understanding the risk of synthetic cdos. Division of Research and Statistics, Federal Reserve Board.

[5] John Hull, Mirela Predescu, and Alan White. The relationship between credit default swap spreads, bond yields, and credit rating announcements. *Journal of Banking and Finance*, 28:2789–2811, November 2004.

[6] John Hull and Alan White. Valuation of a cdo and an n^{th} to default cds without monte carlo simulation. *Journal of Derivatives*, 12(2):8–23, 2004.

[7] Mark S. Joshi. More mathematical finance. `http://www.markjoshi.com/more/`.

[8] Marco Marchioro and Dario Cintioli. Foundations of the statpro simulation model. In *The Simulation Model*, Quantitative Research Series. StatPro website, October 2007. Previews available at `http://marchioro.org`.

[9] Frank Packer and Haibin Zhu. Joshi. Contractual terms and cds pricing. BIS Quarterly Review, March 2005.

[10] Roberto Torresetti, Damiano Brigo, and Andrea Pallavicini. Implied expected tranched loss surface from cdo data. `http://www.damianobrigo.it`.

[11] Oldrich Vasicek. Loan portfolio value. *Risk Magazine*, 15, December 2002.