Portfolio risk management with efficiently simulated scenarios

Marco Marchioro
www.statpro.com
Version 2.0

March 2013*†

Abstract

We describe a method–used, among others, by financial-software firm StatPro–to perform portfolio risk analysis based on a two-tier client/server approach. The risk server computes the numerical simulations of single-asset prices for a wide universe of investable instruments. The risk clients, using the server outcome, compute portfolio cash risk scenarios, stress-test simulations, and bid/ask liquidity spreads. We focus on the risk-client implementation and describe the details needed to compute the different scenario types for a portfolio of heterogeneous assets. It is also shown how it is necessary to treat differently bond-like instruments that always have a positive quote, futures that are settled on a margin account, and swap-like contracts that may have a positive or a negative net-present value. Finally we show how the computation of daily simulations can be used to estimate risk for longer time horizon, even when the financial instrument considered have special bounding constraints.

Contents

*Latest review March 13, 2013.

†The author wishes to thank Davide Borrello for the initial draft of this paper.

1 Introduction

In portfolio risk management we often compute risk measures, stress-test the portfolio, and evaluate the bid/ask liquidity spreads in order to better understand the portfolio behavior under different market conditions. In this paper we describe how to use numerical simulations to perform the portfolio risk management efficiently. Since portfolio returns depend linearly from its constituent assets it is convenient to perform first simulations at asset level and then aggregate them together. However, a complex portfolio can be composed of many different types of financial instruments: bonds, equities, swaps, futures, and other derivatives. Each instrument type can affect the portfolio return in its own different way so that, for example, the contribution of an equity, a bond-futures contract, or a credit-default swap are very different.

Since the numerical simulation of an instrument value depends on a quantitative model, sometimes very complicated and with a high computational cost, it is convenient to dedicate specific hardware for the task. Furthermore many different users may need the simulations of the same asset for different portfolios so that a dedicated *risk server* that only computes simulations is often a good solution. We can then leave the task of portfolio aggregation to separate computational units: the *risk clients*. This dual approach reduces the computational cost since the risk server performs single-asset simulations once for each product, even when the same asset belongs to several portfolios, and the risk clients apply the server pre-computations to the analysis of several portfolios without unnecessary duplicate calculations. In this paper we focus on the computations of the risk client, however we initially summarize the risk server calculations based on the simulation model detailed in reference [3].

The risk server generates three types of simulations:

- the *risk simulations* where the asset prices are computed at a future date using a set of simulated risk factors
- the *stress simulations* provide the asset price in conditions of simulated stressed markets
- the *liquidity simulations* provide possible bid/ask prices due to different bid/ask quotes of the underlying financial variables

An essential requirement to compute a product evaluation is the existence of a quantitative model that provides the instrument value given the appropriate *financial variables*[1]. For example a model used to evaluate a fixed-rate coupon bond is the traditional cash-flow discounting. In this case the risk factors are the interest rates making up the yield curve and the issuer credit spread.

We use the term *pricing function*, hereafter denoted with the symbol $P(\ldots)$, to describe the instrument value depending on the financial variables. We define two categories of risk factors, the stock-like variables (defined more precisely later on) $s_1, s_2, \ldots$, and the rate-like variables (more details later on) $R_k, R_{k+1}, \ldots$, so that the pricing function can be written as

$$P\left(s_1\,, s_2\,, \ldots\,, R_k\,, R_{k+1}\,, \ldots\,; t\right)\,. \tag{1}$$

Given a reference date t we also define the *reference price* P_r as

$$P_r = P\left(\bar{s}_1\,, \bar{s}_2\,, \ldots\,, \bar{R}_k\,, \bar{R}_{k+1}\,, \ldots\,; t\right)\,, \tag{2}$$

where the risk factors $\bar{s}_1, \bar{s}_2, \ldots, \bar{R}_k, \bar{R}_{k+1}$ are observed at the reference date t. We then compute the simulated future prices at a later date t' on a set of shocked risk variables $s'_1, s'_2, \ldots, R'_k, R'_{k+1}, \ldots$

$$P' = P\left(s'_1\,, s'_2\,, \ldots\,, R'_k\,, R'_{k+1}\,, \ldots\,; t'\right)\,. \tag{3}$$

Different ways of shocking the risk variables can be used to obtain different results: risk simulations, stress-test scenarios, or bid/ask spreads.

Since different instrument types need to be treated differently, we distinguish between three kinds of scenario variations. For equities, bonds, and similar products, traders usually think in relative terms. For example an increase of a bond price from 100.00 $ to 102.00 $ corresponds to a variation of 2.00%. In this

[1] In this paper we will also use the term *risk factors* to refer to financial variables

case the *relative-scenario variation* is defined by the ratio

$$\frac{P'}{P_r}\,. \tag{4}$$

For many over-the-counter instruments instead, for instance interest-rate swaps and credit default swaps, the pricing function P provides the net present value. In this case it is convenient to use *absolute-scenario variations* with respect to the given notional amount N_r:

$$\frac{P' - P_r}{N_r}\,. \tag{5}$$

Finally, for futures contracts we are more interested in the variations of the marginal account, hence we define *futures-scenario variations* as

$$\frac{P' - P_r}{P_r}\,. \tag{6}$$

In section 2 we explain more in details the financial motivation behind these choices of scenario-variation types and the technical details of the scenario computations. In particular single-asset risk scenarios are described in section 2.1, stress tests simulations in section 2.2, and bid/ask liquidity spreads in section 2.3. We assume that there is a risk server that computes all price scenarios and that, in some way, they are sent to the risk clients (together with some additional parameters, see details in section 2.4). In section 3.1 we introduce the mathematical formalism necessary to combine scenarios of different products. The core of the paper, from section 3.2 to 3.4, is focused on the methodology of computing portfolio risk, portfolio stress tests, and portfolio liquidity scenarios. In section 4 we define the risk exposure as a measure of the amount of money at risk. In section 5 we describe how to extend the single-period simulations to a different time horizon preserving the instruments bounding properties.

2 Risk-server scenario variations

In this section we describe the simulation techniques used by the risk server and we introduce the precise definitions of risk scenarios, stress-test simulations, and bid/ask liquidity spreads. Since in general the simulated scenarios will be used at a later date than the computation date, we define some *scenario variations* so that the client can adapt the simulations coming from the risk server to market conditions that are slightly changed. In the traditional approach to risk management, born from the equity-asset world, people usually compute price scenarios starting from the ratio between the price at one day and that at the previous day. While this point of view is fine for most traditional assets, it fails for products that can have negative evaluations such as interest-rate swaps. Furthermore since the currency exposure in a futures contract, e.g. an equity futures, is fundamentally different from that on the equity itself, we need to compute different variation types for the former and the latter. In this section we outline the details of scenario-variation computations for three major asset types: bond-like instruments, swap-like products, and futures-like contracts.

2.1 Historical simulations

The computation of risk measures usually requires the future distribution of asset returns. We refer to paper [3] for a detailed description of the simulation technique for future-value distributions and only review the main concepts.

We consider a certain financial instrument with a known pricing function P depending on a finite number of financial variables $s_1, s_2, \ldots R_k, R_{k+1} \ldots$ as in expression (1). Stock-like risk variables, for example stock quotes or foreign-exchange rates, have always a positive value. If the stock price goes from 40.2$ to 46.8$ it is preferable to consider such scenario as a 16.4% increase. Instead rate-like variables, like credit spreads, inflation rates, or swap rates, are usually expressed as a percentage of a notional value and may have negative values (for example credit spreads). If an interest rate goes from 1.30 % to 1.50 % we think of an increase of 0.20 % or 20 basis points. Keeping in mind the difference between stock-like and rate-like risk factors, we define the historically-shocked risk variables to compute the simulated prices, using formula (3).

Considering a stock-like risk factor and a set of quotes on past historical dates, for each date, indexed by i, we compute the risk-factor variations $z^{(i)}$'s as

$$z^{(i)} = \frac{s^{i}_{hist}}{s^{i-1}_{hist}}\,, \tag{7}$$

where s^{i}_{hist} denotes the quote at date i and s^{i-1}_{hist} the quote at the previous available date. Similarly, for rate-like risk factors we write the variations

$$z^{(i)} = R^{i}_{hist} - R^{i-1}_{hist}\,, \tag{8}$$

where R^{i}_{hist} and R^{i-1}_{hist} are, respectively, the risk factors quote at date i and that at the previous available date.

We make the basic assumption that the risk-variable shocks from the reference date to the next business day are distributed as the past daily shocks $z^{(i)}$. In other words we assume the risk variables to be *invariants* according to the definition of reference [4]. As a consequence, given a stock-like risk variable with reference value $\bar{s}_j$, for each date i and shock $z^{(i)}$ we define the shocked risk variable $s^{(i)}_j$ as

$$s^{(i)}_j = \bar{s}_j \cdot z^{(i)}\,. \tag{9}$$

Similarly, given a rate-like risk factor with reference value $\bar{R}_k$ we define the shocked risk variable $R^{(i)}_k$ as

$$R^{(i)}_k = \bar{R}_k + z^{(i)}\,. \tag{10}$$

We then compute the instrument i-th simulated price $P^{(i)}$ at date t', i.e. one business day after the reference date, applying the pricing function P to the shocked risk variables as in equation (3),

$$P^{(i)} = P\left(s^{(i)}_1, s^{(i)}_2, \dots, R^{(i)}_k, R^{(i)}_{k+1}, \dots; t'\right)\,. \tag{11}$$

Depending on the type of financial instrument, for each simulated scenario, we compute a price-scenario variation using the simulated price $P^{(i)}$ and the reference price P_r.

As mentioned in section 1, for bonds, options, and other complex financial instruments with positive monetary value, we define the *relative-scenario variation*, also called *bond-like variation*. Formally, instead of formula (12), we consider a more generic definition depending on the *basis*[2] parameter b:

$$S^{(i)}_r = 1 + b \cdot \left(\frac{P^{(i)} - P_r}{P_r}\right)\,. \tag{12}$$

Note that for a large number of financial products the basis is 1 and equation (12) reduces to formula (4).

The approach of relative scenarios should not be used for assets whose evaluation can become zero or negative, firstly because if the reference price was null then we would not be able to compute the relative scenario. Secondly, the relative variation would not capture the correct sign of the original price variation. For example, consider an interest-rate swap with negative net-present value[3] $P_r = -2.00\$$ and notional value $N_r = 100\$$. Using the bond-like scenarios with basis equal to 1, a simulated evaluation of $P^{(i)} = -1.20\$$ would lead to a positive variation

$$S^{(i)}_r = \frac{P^{(i)}}{P_r} = \frac{-1.20}{-2.00} = 0.6\,,$$

while if the simulated value is small but positive, for instance $P^{(j)} = 1.20\$$, the resulting scenario variation would be negative:

$$S^{(j)}_r = \frac{P^{(j)}}{P_r} = \frac{1.20}{-2.00} = -0.6\,.$$

Therefore for these instrument types we rather consider the cash increase in the net present value normalized by the notional amount N_r:

$$\frac{P^{(i)} - P_r}{N_r}\,.$$

In order to be consistent between different instrument types, we define the *absolute-scenario variations*, also called *swap-like variations*, as

$$S^{(i)}_a = b \cdot \frac{P^{(i)} - P_r}{N_r} \tag{13}$$

[2] More on the basis parameter later on.

[3] For convenience we use the same symbol P_r to denote both the quoted price, e.g. of a bond, the net present value of an OTC contract, and the futures quote. We also freely use the generic terms *price* or *value* for these terms.

where, similarly to the bond-like instruments, we introduced the basis b. Note that when the basis is 1 equation (13) reduces to formula (5). We deal with other unfunded over-the-counter instruments, OTC instruments hereafter, just like we did for swaps.

Finally yet another approach is required when we deal with futures contracts. Since these contracts are settled daily on a margin account we should embed this feature in the scenario variations. Hence, given the basis b, we define the *futures-like variations* as

$$S_f^{(i)} = b \cdot \frac{P^{(i)} - P_r}{P_r} \,. \tag{14}$$

Note that when $b = 1$ this equation reduces to formula (6). Intuitively equation (14) describes the percentage increase, or decrease, of the daily margin account with respect to the market exposure. Consider for example a simulated quoted price $P^{(i)} = 99.30\$$ and a reference price $P_r = 97.20\$$, then

$$S_f^{(i)} = \frac{P^{(i)} - P_r}{P_r} = \frac{99.30 - 97.20}{97.20} = 0.02\,,$$

so that the daily simulated increase in the margin account is given by $P_r \cdot S_f^{(i)} = 1.94\$$. At the end of subsection 4.2 we provide more details to support this choice.

Summarizing, all risk-scenario variations are computed as

$$\begin{aligned} S_r^{(i)} &= 1 + b\,\frac{P^{(i)} - P_r}{P_r} && \text{bond-like,} \\ S_f^{(i)} &= b\,\frac{P^{(i)} - P_r}{P_r} && \text{futures-like,} \\ S_a^{(i)} &= b\,\frac{P^{(i)} - P_r}{N_r} && \text{swap-like.} \end{aligned} \tag{15}$$

Even thought these formulas are different from each other, they all express one simple financial concept: the percentage value variation of the given scenario with respect to the relevant cash measure (either the reference price, for bond-like assets and futures contracts, or the notional ammount for OTC instruments).

Currency-exchange variations When an asset does not have the same currency as the portfolio, we need to apply the currency-exchange scenarios to the asset historical scenarios. Hence, given a currency-exchange rate with current value x and historical values $x_{\text{hist}}^{(i)}$'s, at each date i we can define the *exchange-rate variations* as

$$X^{(i)} = \frac{x_{\text{hist}}^{(i)}}{x_{\text{hist}}^{(i-1)}} \,, \tag{16}$$

where $x_{\text{hist}}^{(i-1)}$ is the latest available quote before date i.

The basis parameter The reason behind the introduction of the additional basis parameter b mainly comes from the world of interest-rate futures. Consider for example the three-months Eurodollar futures, i.e. futures on the three-month USD LIBOR rate. These contracts are quoted as 100 minus the annual eurodollar rate and settled on a margin account. For example a quote value of 96.00 means an interest rate of 4.00%. A change of the interest rate of minus 10 basis points is reflected in a change of 0.01 of the quoted price, for example from 96.00 to 96.01; when the futures contract has a notional of $1,000,000\,\$$ the gain or loss (depending if the contract is long or short) can be computed as

$$1,000,000\,\$ * \frac{90}{360} * \frac{0.10}{100} = 250\,\$.$$

Since the scenario variation is defined to provide the *value* increment in a scenario, with respect to the notional amount, we need to correct definition (5) by introducing the basis b. In this example of three-month interest-rate futures we define $b = 1/4$. In table 1 we summarize when the basis should be left user definable for cases similar to interest-rate futures contracts.

2.2 Stress tests

For stress-test scenarios we use an approach similar to risk, however, instead of historically shocked risk variables we consider some risk-factor displacements corresponding to specific stressed markets. Each stock-like $s_j^{(s)}$ or rate-like $R_k^{(s)}$ shocked risk factor is then computed applying a stress displacement to its reference value, similarly to formulas (9) and (10). For each stress index s, the simulated evaluation $P^{(s)}$ is computed as

$$P^{(s)} = P\left(s_1^{(s)}, s_2^{(s)}, \ldots, R_k^{(s)}, \ldots; t\right), \qquad (17)$$

similarly to the risk scenarios of equation (11). Note that the stressed price is computed at the reference date t.

Consider as an example a vanilla equity option with a certain strike. The pricing function P in this case is the Black-Scholes-Merton formula depending on three risk factors: the interest rate, the volatility, and the asset spot price (we neglect dividends). Assume the option reference price is $P_r = 12.80\$$, computed using an interest rate R_r, an equity volatility σ_r, and an equity spot price s_r. A possible computation for the stressed price is

$$P^{(s)} = P\left(R', \sigma', s'\right) = 13.75\,\$, \qquad (18)$$

computed with a stressed interest rate $R = R_r - 0.25\,\%$, a stressed volatility $\sigma_r(1{+}20\%)$, and a stressed equity price $s' = s_r(1 - 30.0\,\%)$. For bond-like instruments the option stress scenario variation should be computed as

$$S_r^{(s)} = \frac{P^{(s)}}{P_r} = \frac{13.75}{12.80} = 1.0742\,. \qquad (19)$$

In general, using definitions similar to the risk case, we introduce three types of *stress-scenario variations*:

$$\begin{aligned} S_r^{(s)} &= 1 + b\,\frac{P^{(s)} - P_r}{P_r} && \text{bond-like,} \\ S_f^{(s)} &= b\,\frac{P^{(s)} - P_r}{P_r} && \text{futures-like,} \\ S_a^{(s)} &= b\,\frac{P^{(s)} - P_r}{N_r} && \text{swap-like.} \end{aligned} \qquad (20)$$

Setting $b = 1$ we obtain formulas corresponding to expressions (4), (5), and (6).

2.3 Bid/ask liquidity spreads

Here we briefly recall the approach introduced in reference [1] to compute market bid/ask liquidity spreads. Of all the different definitions of liquidity we focus on the *market risk-factor-induced bid/ask liquidity*, i.e. the widening of the bid/ask spread for an instrument induced by the bid/ask widening of its risk factors. In turn, the widening of the asset bid/ask spread causes a market loss in selling (or buying) certain assets that can be thought as one type of *liquidity risk*. We assume this loss to be generated by the difference between the price at which the financial asset is marked and the price at which it can be sold, so that we can estimate the difference between the ask price and the bid price.

In brief: we model the liquidity-scenario simulations of a financial instrument as the linearized sum of variations coming from the risk factors. We select the risk variables with variations that lead to an increase of the product value and those that imply a decrease. We use the first set to simulate the ask prices and the second one to simulate the bid prices.

1) In describing the 3 liquidity scenarios I would be more specific in defining what they aim at representing. This has created much confusion with clients and we need to be precise on the fact that it is up to the user to decide which of the 3 "standard" scenarios applies to the current situation;

The induced liquidity bid/ask spreads vary greatly in time and across asset classes. Indeed, while the view of one asset manager is that a certain market is very stressed at a given moment, it is entirely possible that another investor feels the market to be in a quite state. In computing the bid/ask simulations on the risk server we need to provide at the same time scenario for very different market conditions and leave up to the user which one to use. Therefore the risk server daily computes bid/ask liquidity simulations for at least three different market conditions: normal markets, stressed markets, and highly-stressed markets.

More precisely, for each market condition we consider a set of displacements $L = \{\pm l_1, \pm l_2, \ldots\}$, where each displacement l_k or $-l_k$ refers to the k-th risk variable, and we compute the simulated prices:

$$
\begin{aligned}
P_1^L &= P\,(\,s_1\cdot(1+l_1)\,,s_2\,,\dots\,,R_k\,,R_{k+1}\,,\dots)\ ,\\
P_1^S &= P\,(\,s_1\cdot(1-l_1)\,,s_2\,,\dots\,,R_k\,,R_{k+1}\,,\dots)\ ,\\
&\vdots\\
P_k^L &= P\,(\,s_1\,,s_2\,,\dots\,,R_k+l_k\,,R_{k+1}\,,\dots)\ ,\\
P_k^S &= P\,(\,s_1\,,s_2\,,\dots\,,R_k-l_k\,,R_{k+1}\,,\dots)\ ,\\
&\vdots
\end{aligned}
$$

where all evaluations are performed at the reference date t. We then create two types of scenarios, respectively *ask* and *bid*, that can be used when the asset in the portfolio is, respectively, *long* and *short*. In the case of *long* scenarios, for each stock-like or rate-like risk variable indexed by k, we select the smallest among P_k^L, P_k^S or P_r, and we mark such price as P_k^-, so that

$$P_k^- = \min\left(P_k^L, P_k^S, P_r\right)$$

and we always have

$$P_k^- \le P_r\,. \tag{21}$$

Therefore when an asset is long in a portfolio we always simulate a sell at a price lower than the reference. On the other hand, for assets that will be simulated as short in the portfolio we need instead to consider a buy (i.e. an exit from the short position) with a price that is higher than the market price. Hence we need to define a *short simulated price* P_k^+ so that,

$$P_k^+ \ge P_r\,.$$

This can be accomplished by choosing

$$P_k^+ = \max\left(P_k^L, P_k^S, P_r\right)\ . \tag{22}$$

We then put together the simulations coming from each risk factor. The easiest way to do this is to sum up all long and short contributions from each risk factor. Hence, for the given market condition, the simulated bid prices are given by

$$P^- = P_r\left[1+\left(\frac{P_1^- - P_r}{P_r}\right)+\left(\frac{P_2^- - P_r}{P_r}\right)+\dots\right]$$

for bond-like instruments and futures-like contracts, and

$$P^- = P_r + \left(P_1^- - P_r\right) + \left(P_2^- - P_r\right) + \dots$$

for swap-like products. Similarly, the short prices, i.e. the simulated ask prices, are computed as

$$P^+ = P_r\left[1+\left(\frac{P_1^+}{P_r}-1\right)+\left(\frac{P_2^+}{P_r}-1\right)+\dots\right]$$

for bond-like instrument and futures-like contracts, while for swap-like products we have

$$P^+ = P_r + \left(P_1^+ - P_r\right) + \left(P_2^+ - P_r\right) + \dots\ .$$

Notice that we always obtain $P^- \le P_r$ and $P^+ \ge P_r$.

Notice that since we assume a zero bid/ask spread for currency-exchange rates, bond-like instruments and futures-like contracts behave similarly. For each instrument the long-side liquidity scenarios are computed similarly to the risk and the stress scenarios using the bid/ask simulated haircuts S^-:

$$
\begin{aligned}
S_{(r,f)}^- &= b\cdot\left[\frac{P_r - P^-}{P_r}\right]_0^1 && \text{bond-like and futures-like,}\\
S_a^- &= b\cdot\left[\frac{P_r - P^-}{N_r}\right]_0 && \text{swap-like,}
\end{aligned}
\tag{23}
$$

where we apply a minimum and a maximum value for numerical reason. In definitions (23) we introduced the *brackets bounding notation* to simplify some of the following expressions. Given a quantity Y with cap C or floor F, we denote by

$$
\begin{aligned}
[\,Y\,]^C &= \min\,[C\,;Y\,]\ , && (24)\\
[\,Y\,]_F &= \max\,[F\,;Y\,]\ , && (25)\\
[\,Y\,]_F^C &= \min\,[C\,;\max\,(F\,;Y)]\ . && (26)
\end{aligned}
$$

Similarly to the long side, the short-side scenario variations (or haircuts) S^+ are defined as,

$$
\begin{aligned}
S_{(r,f)}^+ &= b\cdot\left[\frac{P^+ - P_r}{P_r}\right]_0 && \text{bond-like and futures-like,}\\
S_a^+ &= b\cdot\left[\frac{P^+ - P_r}{N_r}\right]_0 && \text{swap-like.}
\end{aligned}
\tag{27}
$$

Note that all the haircuts are always positive by construction.

2.4 Server/client data exchange

So far we have described how the server-side computations are performed. Given a generic instrument, the risk server computes the reference price P_r, the historical risk-scenario variations $S^{(i)}$'s the stress-scenario variations $S^{(s)}$'s, the liquidity bid/ask variations $S^{\pm}$'s for each market condition, and then makes the data available to all risk clients . As explained in the previous subsections these variations are computed using equations (15), (20), (23), and (27). In order to avoid cumbersome notation, from now on we omit the dependency on the scenario type (r, f, a) when not necessary and we introduce the vector notation for risk scenarios:

$$\mathbf{S} = \left\{ S^{(i)} \text{ for } i = 1, 2, \ldots \right\} . \tag{28}$$

For each product the risk server transfers to all clients all scenario variations, the historical dates associated to the risk scenarios, and the price factor coming from the instrument term sheet.

The price factor The price factor f is a multiplier needed to compute the cash variations from the price variations. The instruments are split in groups, called product types, having similar nature and conventions. The price factor value depends on the product type and can be either fixed or user defined.

For instance, a bond quote is typically expressed as a percentage of par value, usually 100.00. Therefore bonds–other than inflation-linked bonds–have a price factor $f = 0.01$. The inflation-linked bonds have a price factor, called indexation coefficient, given by the ratio between the quote of the underlying inflation index at the maturity date and its quote at the issue date. Instruments defined on unitary notional, such as warrants, certificates, or OTC products, require a price factor $f = 1$. Finally products defined over lots allow a user-definable price factor. See table 1 for the list of available price factors and product types.

3 Portfolio cash scenarios

In the traditional handling of portfolio risk management one considers the combined effect of financial markets on several assets within an investment portfolio. Because of diversification effects the portfolio may have very different risk figures, stress-test simulations, and bid/ask liquidity spreads from its constituting instruments when taken separately. In order to compute the portfolio simulations, the risk clients combine the single-asset scenarios computed by the risk server to perform portfolio-level analysis.

Risk-client parameters and variables The server simulations are usually computed at an earlier time than when they are used by the clients. Each risk client therefore needs to adapt those simulations to the latest market data available. This is possible because the risk server transfers to the risk clients the simulation variations, as computed in the previous section, instead of the evaluated simulation themselves. In order to measure an up-to-date portfolio risk, bid/ask liquidity spreads, or perform stress tests, the risk-client users need to provide the latest market data, for each asset, as follows:

- for equities and bonds the quoted user price P_u, for futures contracts the market user quote P_u, for swaps the unitary net-present value p_u ; all in the asset currency
- the exchange rate x between the asset currency and the portfolio currency (when an asset and the portfolio have the same currency we set $x = 1$)
- the *quantity* q: for equities the number of shares, for OTC contracts the notional amount, for futures the number of lots, and so on and so forth (see table 1 for more details)

Note that in the specific case of options (equity options, index options, bond options) the simulated server scenarios can widely vary from day to day according to the option moneyness. In this case a misalignment of the server evaluation with the user price can lead to significantly discrepancies in the computation of risk

Product type	**Price factor**	**Basis**	**Quantity**
Equities	1	1	units
Mutual funds	1	1	units
Indices	1	1	units
Bonds	0.01	1	nominal
Inflation Bonds	variable	1	nominal
Korean Bonds	0.0001	1	nominal
New style inflation bonds	indexation coefficient	1	nominal
Warrant/Certificates	1	1	units
Futures/FX forwards	user defined	user defined	lots
Options	user defined	user defined	lots
Liquidity	1	1	nominal
Structured bonds	0.01	1	nominal
Generic OTC	1	1	quantity/notional
OTC options	1	1	quantity/notional
Single-leg futures	user defined	user defined	lots

Table 1: Product types and parameters of StatPro Risk Server

measure and the user price should be set equal to the server model price: i.e. $P_u = P_r$.

The additional requirements for the client computations, usually available from the server side, are

- to compute portfolio risk, the risk variations $S^{(i)}$'s given by formulas (15), together with the set of historical dates they refer to
- to compute the portfolio stress scenarios, the stress variations $S^{(s)}$'s computed by the risk server using definitions (20)
- to compute portfolio bid/ask liquidity spreads, the liquidity scenarios $S^{\pm}$'s, for each market condition, defined in equations (23) and (27)
- finally, the price factor f and the basis b

The methodology we are about to describe is efficient because does not require the full pricing functions to be available on the risk clients.

Summary of portfolio scenario computations First we give here a bird-eye view of the steps needed to compute the cash portfolio scenarios and later we describe those steps in great details.

1. For each instrument in a currency different from the portfolio currency, we convert its reference evaluation and its scenario variations in the portfolio currency
2. We convert the scenario variations for each instrument into cash scenarios
3. We scale the cash scenarios by the asset quantity in the portfolio
4. For each historical-scenario date, we compute the portfolio cash increase (or decrease) by summing up the cash variations of each asset (considering appropriately the missing scenarios)

Using the portfolio scenarios we can then compute a number of risk measures as explained in reference [3].

3.1 Viable historical scenario dates

Since financial markets follow different holiday schedules two instruments may have quotes available at different date sets. Therefore, in order to combine simulations for different products in the same portfolio, we need to account for the different set of historical dates for which simulations can computed. For this reason we introduce the concept of the *inadmissibility mask*.

The inadmissibility mask Consider an instrument and its set of risk-scenario variations $S^{(i)}$'s, either of bond-like type, futures-like type, or swap-like type. First of all we introduce the vector notation so that the bold face, e.g. $\mathbf{S}$, denotes the collection of $S^{(i)}$ for all simulation indexes i's. In this notation all scenarios are considered at once and each element of $\mathbf{S}$ is associated to an historical risk-variable shock at a past date t_i. The dates t_i's are not necessarily consecutive, more often than not, some dates are missing either because instruments are not quoted on some financial markets or because, on other dates, there were some missing data that prevented the simulation from being computed. Furthermore, in order to limit the number of computations, we define a range of dates for which all instruments may have scenarios. For example we may limit ourselves to simulations over the last two years. Also, given a reference date t, scenarios could in principle be added by the client-side for dates later than t. Therefore we define a number of *look-back days* d_{back} and a number of *forward days* d_{forw} so that we consider the universe of dates going back d_{back} days from the reference date and going forward d_{forw} days. More precisely, we consider the *earliest date* d_{1st}, or first date, and the *last date* d_{last} defined as

$$d_{\text{1st}} = t - d_{\text{back}} \qquad \text{and} \qquad d_{\text{last}} = t + d_{\text{forw}} \,.$$

For example when we are interested in about two years of historical scenarios a good choice to determine the earliest and the latest dates is given by

$$d_{\text{back}} = 800 \qquad \text{and} \qquad d_{\text{forw}} = 21 \,.$$

We then define the inadmissibility mask $\mathbf{M}$ to be the vector of boolean values marking the dates where the scenarios were *not computed*. More explicitly, at a certain date for a given product, the mask has the *truth* value when the scenario is not computed, in this case the date is an *inadmissible date*, otherwise it is *false* and the date is *viable*.

Default values for inadmissible dates To compute portfolio scenarios we want the risk variations for each instrument to be defined between $d_{1\text{st}}$ and d_{last}. Therefore if the inadmissibility mask is false for a certain date t_i we assign to the risk scenario $S^{(i)}$ a default value. These default values should be chosen so that they provide the least variation in the simulated evaluations. Therefore we define the default values

$$S^{(i)} = \begin{cases} 1 & \text{bond-like,} \\ 0 & \text{futures -like and swap-like;} \end{cases} \tag{29}$$

and

$$X^{(i)} = 1\,, \tag{30}$$

for currency-exchange variations.

3.2 Portfolio cash risk scenarios

In this subsection we show how to compute the cash risk scenarios for different assets and aggregate them to obtain portfolio cash scenarios. Given the scenario variation vector $\mathbf{S}^j_{(r,f,a)}$ for each asset indexed by j, we consider the value simulations

$$\begin{cases} P_u \cdot \mathbf{S}^j_r & \text{bond-like,} \\ P_u \cdot \left(1 + \mathbf{S}^j_f\right) - P_u & \text{futures-like,} \\ p_u + \mathbf{S}^j_a & \text{swap-like.} \end{cases} \tag{31}$$

For futures-like simulations P_u has been subtracted from the scenario because the value is not in the quoted price P_u, however, in the associated margin account. Note that

$$P_u \cdot \left(1 + \mathbf{S}^j_f\right) - P_u = P_u \cdot \mathbf{S}^j_f \tag{32}$$

so that the price simulations for bond-like products and futures-like contracts are the same.

In these and in the following operations between vectors and scalars we assume that each element of the product between a vector and a scalar is the product of the vector element and the scalar; each element of the sum of a vector with a scalar is the sum of the vector element with the scalar; finally for future uses, each element of the product between two vectors is defined as the product of the first-vector element and the second-vector element.

Recall that in this subsection the index j refers to different assets and that should not be confused with the scenario index i. In any case to minimize the possibility of confusion we try, when possible, to avoid writing the scenario index i and to use the vector notation introduced in subsection 2.4.

Currency conversion of historical scenarios When an asset is in a different currency from that of the portfolio, not only we need to consider the exchange rate between the two currencies, we need also to account for the exchange-rate scenarios. Omitting the asset index j, for each asset we consider the vector of exchange-rate variations $\mathbf{X}$ with elements defined in equation (16), then if x is the exchange rate between the asset currency and the portfolio currency, with $x = 1$ possibly when the two are the same, the exchange-rate scenarios are given by

$$x^{(i)} = x \cdot X^{(i)} \qquad \left(\text{or } \mathbf{x} = x \cdot \mathbf{X}\right), \tag{33}$$

i.e. by the vector of exchange-rate variations multiplied by the current exchange rate. Applying the exchange-rate scenarios to the value scenarios of expression (31) we obtain

$$\begin{cases} \left(P_u \cdot \mathbf{S}_{(r,f)}\right) \cdot (x \cdot \mathbf{X}) & \text{bond-like and futures-like,} \\ (p_u + \mathbf{S}_a) \cdot (x \cdot \mathbf{X}) & \text{swap-like,} \end{cases} \tag{34}$$

where, again, operations on vectors are performed element by element.

Meaningful scenarios can be computed from expression (34) only within the range of viable dates.

Hence suppose that the simulations for the asset were performed from an earlier date $d_{\min}$ and a later date $d_{\max}$, assuming $d_{\min} \geq d_{1st}$ and $d_{\max} \leq d_{last}$, also the exchange-rate variations to be defined between an earlier date $d^x_{\min}$ and a later date $d^x_{\max}$, then scenarios defined in expression (34) should be considered valid only between a minimum date

$$\bar{d}_{\min} = \max(d_{\min},\, d^x_{\min})\,, \tag{35}$$

and a maximum date

$$\bar{d}_{\max} = \min(d_{\max},\, d^x_{\max})\,. \tag{36}$$

Between these dates the scenarios can be computed with the help of equation (29) and (30) for the missing scenarios. Furthermore denoting with $\mathbf{M}$ the inadmissibility mask for the asset variations and $\mathbf{M}_x$ that of its exchange rates, the scenarios of expression (34) should have a mask given by

$$\overline{\mathbf{M}} = \mathbf{M} \,\&\, \mathbf{M}_x\,, \tag{37}$$

where the operator & reppresents the element-by-element Boolean *and* operator. In other words scenarios (34) are viable if they are either for $\mathbf{S}$ or $\mathbf{X}$ and for dates that are earlier than the maximum date for both variation vectors and later than the minimum date for both variation vectors.

Asset cash simulations Certainly different assets are held in the portfolio in different quantities. In an earlier section we have already defined for each asset a quantity q as the number of shares of a stock in a portfolio, the bond nominal amount, the notional amount of a swap, and so on and so forth. Taking into account the quantity q of each asset, omitting again the asset index j, the client-side simulated values of expression (34) become

$$\begin{cases} q\cdot\left[P_u\cdot \mathbf{S}_{(r,f)}\cdot x\cdot \mathbf{X}\right] & \text{bond-like and futures-like,}\\[2ex] q\cdot[p_u+\mathbf{S}_a]\cdot x\cdot\mathbf{X} & \text{swap-like.}\end{cases} \tag{38}$$

In order to obtain the cash scenarios, also known as P&L scenarios, for each asset we need to subtract the *equilibrium* evaluation

$$\begin{cases} P_u\cdot x\cdot q & \text{bond-like,}\\[2ex] 0 & \text{futures-like,}\\[2ex] p_u\cdot x\cdot q & \text{swap-like.}\end{cases} \tag{39}$$

Furthermore, as seen at the end of section 2, to transform the price variations into cash variations we need to multiply all scenarios by the price factor f. Therefore, the cash scenarios for each asset in the portfolio can be computed as

$$\begin{aligned} \mathbf{R}_r &= (\mathbf{S}_r\cdot\mathbf{X}-1)\cdot P_u\, x\, q\, f && \text{bond-like,}\\ \mathbf{R}_f &= (\mathbf{S}_f\cdot\mathbf{X})\cdot P_u\, x\, q\, f && \text{futures-like,}\\ \mathbf{R}_a &= [(p_u+\mathbf{S}_a)\cdot\mathbf{X}-p_u]\, x\, q\, f && \text{swap-like,} \end{aligned} \tag{40}$$

where, again, we omitted the asset index j.

Portfolio cash simulations We can write the cash scenarios of equations (40) for each asset indexed by j as

$$\begin{aligned} \mathbf{R}^j_r &= \left(\mathbf{S}^j_r\cdot\mathbf{X}^j-1\right)\cdot P^j_u\, x^j\, q^j\, f^j && \text{bond-like,}\\ \mathbf{R}^j_f &= \left(\mathbf{S}^j_f\cdot\mathbf{X}^j\right)\cdot P^j_u\, x^j\, q^j\, f^j && \text{futures-like,}\\ \mathbf{R}^j_a &= \left[\left(p^j_u+\mathbf{S}^j_a\right)\cdot\mathbf{X}^j-p^j_u\right] x^j\, q^j\, f^j && \text{swap-like,} \end{aligned} \tag{41}$$

Recall that these scenarios are viable for all scenario dates later than $\bar{d}^j_{\min}$, earlier than $\bar{d}^j_{\max}$, and subject to the inadmissibility mask $\overline{M}^j$. The portfolio cash scenarios, also known as P&L scenarios, can be computed simply as the sum of cash scenarios for each asset in the portfolio:

$$\mathbf{R}^{\text{port}} = \sum_{j=1}^{n}\mathbf{R}^j = \mathbf{R}^1+\mathbf{R}^2+\ldots+\mathbf{R}^n\,. \tag{42}$$

This definition, together with expressions (41) , is one of the core results of this paper. However we still need to describe the range of dates for which this expression is viable. Consistently with the approach of previous paragraphs, we define the portfolio minimum viability date to be the highest among the currency-adjusted asset minimum dates,

$$d^{\text{port}}_{\text{min}} = \max\left(\bar{d}^1_{\text{min}}, \bar{d}^2_{\text{min}}, \ldots, \bar{d}^n_{\text{min}}\right) , \qquad (43)$$

and likewise for the portfolio maximum date:

$$d^{\text{port}}_{\text{max}} = \min\left(\bar{d}^1_{\text{max}}, \bar{d}^2_{\text{max}}, \ldots, \bar{d}^n_{\text{max}}\right) . \qquad (44)$$

Similarly we define the portfolio inadmissibility mask as

$$\mathbf{M}^{\text{port}} = \overline{\mathbf{M}}^1 \,\&\, \overline{\mathbf{M}}^2 \,\&\, \ldots \,\&\, \overline{\mathbf{M}}^n , \qquad (45)$$

so that a portfolio scenario date between $d^{\text{port}}_{\text{min}}$ and $d^{\text{port}}_{\text{max}}$ is viable if it is so for at least one asset.

Portfolio risk measures The main purpose of the cash scenario computation is to compute portfolio risk measures. We leave the details of the subject to the specialized literature and only describe the main concepts. In loose terms a risk measure is a number that describes certain aspects of a distribution of possible future portfolio values. Popular risk measures are value at risk, volatility, expected shortfall, and so on and so forth.

In the environment or risk simulations described in this paper we define a risk measure as a function of portfolio cash risk simulations $\mathbf{R}^{\text{port}}$,the inadmissibility mask $\mathbf{M}^{\text{port}}$, the minimum and maximum viable dates $d^{\text{port}}_{\text{min}}$ and $d^{\text{port}}_{\text{max}}$, i.e.

$$\mathcal{R} = \mathcal{R}\left(\mathbf{R}^{\text{port}}, \mathbf{M}^{\text{port}}, d^{\text{port}}_{\text{min}}, d^{\text{port}}_{\text{max}}\right) . \qquad (46)$$

This definition of risk measure is quite general and should be specified further in practical cases.

3.3 Portfolio cash stress scenarios

In this subsection we define the portfolio cash stress scenarios given the stress variations $S^{(s)}$'s computed by the risk server, see equation (20), and the market evaluation, either P_u or p_u, provided by the user. Since stress tests depend on a set of given risk-factor displacements the definition of mask, introduced in the last subsections, is not required and we can focus on the stress scenarios independently from each other.

Given a portfolio asset and a stress label s the asset cash stress scenarios $R^{(s)}$ can be computed similarly to the risk case, described by equation (40), as

$$\begin{aligned} R^{(s)}_r &= \left[S^{(s)}_r\, S^{(s)}_x - 1\right] P_u\, x\, q\, f && \text{bond-like,} \\ R^{(s)}_f &= \left[S^{(s)}_f\, S^{(s)}_x\right] P_u\, x\, q\, f && \text{futures-like,} \\ R^{(s)}_a &= \left[\left(p_u + S^{(s)}_a\right) S^{(s)}_x - p_u\right] x\, q\, f && \text{swap-like.} \end{aligned} \qquad (47)$$

The portfolio stress-test scenarios are given by the sum of the stress scenarios for each asset, however when stress scenarios are not available for a certain instrument, we use instead the null variations defined as

$$\begin{aligned} R^{(s)}_r &= \left[S^{(s)}_x - 1\right] P_u\, x\, q\, f && \text{bond-like} \\ R^{(s)}_f &= 0 && \text{futures-like,} \\ R^{(s)}_a &= \left[S^{(s)}_x - 1\right] p_u\, x\, q\, f && \text{swap-like,} \end{aligned} \qquad (48)$$

to be compared with equations (29).

3.4 Portfolio cash liquidity scenarios

For each market liquidity condition we compute the cash bid/ask spreads of a given portfolio from the simulated market bid/ask variations $S^{\pm}$ defined in equations (23) and (27). We make the reasonable assumption that currency-exchange rates have negligible bid/ask spreads so that there are no bid/ask scenarios associated with the currency-exchange risk factor. For the same reason the liquidity scenarios, defined in section 2.3, satisfy the same equations for bond-like instruments and futures-like contracts.

In order to compute the bid/ask simulations for an asset in a portfolio we can use the same arguments that led to expressions (40) for risk and (47) for stress scenarios, however without exchange-rate scenarios. Therefore for each asset and market condition, the *long* cash bid/ask scenarios R^- are given by

$$\begin{aligned} R^-_{(r,f)} &= S^-_{(r,f)} P_u \, x \, q \, f && \text{bond-like and futures-like,} \\ R^-_a &= S^-_a \, x \, q \, f && \text{swap-like,} \end{aligned} \qquad (49)$$

for any simulated bid scenario variation S^-. The *short* cash liquidity scenarios R^+ are given by

$$\begin{aligned} R^+_{(r,f)} &= S^+_{(r,f)} P_u \, x \, q \, f && \text{bond-like and futures-like,} \\ R^+_a &= S^+_a \, x \, q \, f && \text{swap-like,} \end{aligned} \qquad (50)$$

for any simulated ask scenario variation S^+. Recall that $S^\pm$ is always constrained to be null or positive, i.e.,

$$S^\pm \geq 0 \,, \qquad (51)$$

furthermore we also have,

$$S^\pm \leq 1 \,. \qquad (52)$$

for bond-like instruments and futures-like contracts. The portfolio long or short liquidity scenarios are given by the sum of the long or short liquidity scenarios for each asset assuming a null scenario for those products that were not computed from the risk server.

3.5 Weekly risk scenarios

In subsection 2.1 we have shown how the risk server computes the historical-simulation scenarios for a *single period*, usually from the reference date to the next business day (see equations (2) and (3)). In this section we define the *weekly historical scenarios* computed on a time horizon of five consecutive business days. The given procedure works well for linear pricing functions, i.e. equity-like assets such as funds, currency-exchange rates, and indexes; it does not work too well for non-linear pricing functions such as those describing bonds, options, and so on and so forth.

To understand the basic concepts consider, for example, a stock with reference price 102.30 \$ and five historically simulated prices $P^{(1)}, \ldots, P^{(5)}$ corresponding to a certain past week from Monday to Friday. Suppose the single-period, i.e. daily, simulations are given by the following scenario prices

$$\begin{aligned} P^{(1)} &= 103.30\,\$ \\ P^{(2)} &= 102.55\,\$ \\ P^{(3)} &= 102.90\,\$ \\ P^{(4)} &= 101.95\,\$ \\ P^{(5)} &= 102.90\,\$. \end{aligned}$$

In the standard historical-simulation framework we interpret these values as five possible instrument prices all happening the date following the reference date. However, for equities we have a linear pricing function and using equation (7) and (12), with b=1, we obtain

$$S^{(i)}_r = \frac{P^{(i)}}{P_r} = \frac{s^i_{hist}}{s^{i-1}_{hist}} \,. \qquad (53)$$

Using this result, we can use the five consecutive scenario variations $S^{(1)}_r, \ldots, S^{(5)}_r$ to compute the weekly variation $W^{(1)}_r$,

$$\begin{aligned} W^{(1)}_r &= S^{(1)}_r \cdot S^{(2)}_r \cdot S^{(3)}_r \cdot S^{(4)}_r \cdot S^{(5)}_r \qquad (54) \\ &= \frac{s^1_{hist}}{s^0_{hist}} \cdot \frac{s^2_{hist}}{s^1_{hist}} \cdot \frac{s^3_{hist}}{s^2_{hist}} \cdot \frac{s^4_{hist}}{s^3_{hist}} \cdot \frac{s^5_{hist}}{s^4_{hist}} \\ &= \frac{s^5_{hist}}{s^0_{hist}} \,, \end{aligned}$$

In the given example we have $W^{(1)}_r \simeq 0.99$.

More in general, given a set $\mathcal{D}$ of N daily scenarios $S^{(i)}$ referring to N past dates, we discard all the dates earlier than the first Monday and all the dates later than the last Friday (so that consider only full weeks in $\mathcal{D}$ from Monday to Friday). Then within each week we compose the scenarios as in definition (54) assuming a value of 1 for all missing scenarios.

The generalization of the equity-like scenarios is straightforward for all bond-like instrument (however one can doubt its applicability):

$$W^{(j)}_r = S^{(j)}_r \cdot S^{(j+1)}_r \cdot \ldots \cdot S^{(j+4)}_r \,. \qquad (55)$$

Similarly, for instruments with swap-like or futures-like scenarios a weekly scenario $W^{(j)}$, depending on $S^{(j+0)}$ to $S^{(j+4)}$ daily variations, can be defined as

$$W_f^{(j)} = \left(S_f^{(j)} + 1\right) \cdot \left(S_f^{(j+1)} + 1\right) \cdot \ldots \cdot \left(S_f^{(j+4)} + 1\right) - 1 \tag{56}$$

for futures-like instruments and

$$W_a^{(j)} = S_a^{(j)} + S_a^{(j+1)} + \ldots + S_a^{(j+4)} \tag{57}$$

for swap-like products. For swap-like or futures-like scenarios we fill the missing data assuming the null default values as defined in equation (29). Notice that the futures-like weekly scenarios are the product of the *bond-like* daily price variations, then converted into futures-like scenarios. This is equivalent to have futures contracts with a weekly-settled marginal account.

4 Risk exposure

An internet search for the term risk exposure may return results like[4]: *"the quantified potential for loss that might occur as a result of some activity."* However this definition does not help much when a quantitative answer is needed. Indeed there are many different quantitative definitions of risk exposure depending on the specific market we look at. For example, for a bond one may use either the nominal redemption amount or the same amount multiplied by the clean price and divided by 100; for a swap one may use the notional swap amount, its net present value, or something else. Intuitively, for each asset in a portfolio we define the risk exposure to be the *cash equivalent subject to risk*.

To look for a definition that is valid across asset classes we compare the expressions (40), (47), (49), and (50), that we derived for risk simulations, stress tests, and bid/ask-liquidity spreads. In all equations one group of parameters stands out:

$$\begin{aligned} E_{(r,f)} &= P_u \cdot x \cdot q \cdot f && \text{for bond-like and futures-like,} \\ E_a &= x \cdot q \cdot f && \text{for swap-like}\,. \end{aligned} \tag{58}$$

We call the parameter E thus defined to be the *risk exposure* of the asset in the portfolio.

For example for a bond-like instrument the risk exposure is the product of the quoted price in the portfolio currency, the asset quantity, and the price factor. For instance consider a bond, in the portfolio currency, with market price[5] $P_u = 103.40$, a price factor $f = 0.01$ (see table 1), and a nominal amount $q = 1,000,000\$$, using equation (58) we can compute the exposure as

$$E_r = 103.40 \cdot 1,000,000\$ \cdot 0.01 = 1,034,000.00\$\,.$$

For swaps the risk exposure is similarly computed as the notional, also denoted with the symbol q, multiplied by the price factor and by the portfolio/asset exchange rate.

4.1 Unitary cash scenarios

By construction, the definitions (58) of risk exposure can be used to simplify the cash-scenario formulas of risk, stress-test simulations, and bid/ask liquidity spreads. For risk we define the *unitary cash risk scenarios* of an asset in a portfolio as

$$\begin{aligned} \mathbf{U}_r &= \mathbf{S}_r \cdot \mathbf{X} - 1 && \text{bond-like,} \\ \mathbf{U}_f &= \mathbf{S}_f \cdot \mathbf{X} && \text{futures-like,} \\ \mathbf{U}_a &= (p_u + \mathbf{S}_a) \cdot \mathbf{X} - p_u && \text{swap-like.} \end{aligned} \tag{59}$$

Likewise, for each stress test indexed by s, we define

$$\begin{aligned} U_r^{(s)} &= S_r^{(s)} \cdot X^{(s)} - 1 && \text{bond-like,} \\ U_f^{(s)} &= S_f^{(s)} \cdot X^{(s)} && \text{futures-like,} \\ U_a^{(s)} &= \left(p_u + S_a^{(s)}\right) \cdot X^{(s)} - p_u && \text{swap-like,} \end{aligned} \tag{60}$$

to be the *unitary cash stress-test scenarios* for an asset in a portfolio. Finally for bid/ask liquidity spreads, in agreement with equations (49), and (50), we write

[4] See, e.g. `http://www.businessdictionary.com/definition/risk-exposure.html`

[5] Either clean price or dirty price can be used as market price, as long as this choice is consistent for all bonds.

the *unitary cash liquidity scenarios* for a given market condition as

$$U^{-} = S^{-} \qquad \text{and} \qquad U^{+} = S^{+}\,, \qquad (61)$$

for all instrument types. The simplicity of this definition derives from neglecting exchange-rate bid/ask spreads. Unitary cash scenarios are dimensionless and can be thought as the cash scenarios for one unit of exposure.

Simplified formulas for cash scenarios Using the definition of risk exposure and that of unitary cash scenarios we can write the expression for an asset cash risk scenario vector, namely formulas (40), as

$$\mathbf{R} = \mathbf{U} \cdot E\,, \qquad (62)$$

for all instrument types. Similarly we can write the stress-test cash scenarios for an asset in a portfolio, see equation (47), as

$$R^{(s)} = U^{(s)} \cdot E\,, \qquad (63)$$

Finally, for any given market condition, the bid/ask liquidity cash scenarios of an asset in a portfolio, see formulas (49) and (50), can be computed as

$$R^{-} = U^{-} \cdot E \qquad \text{and} \qquad R^{+} = U^{+} \cdot E\,, \qquad (64)$$

for all variation types.

Using these equations we can also simplify the formulas for portfolio cash scenarios. Hence, denoting by E^j the risk exposure of the j-th asset, given its unitary risk scenarios $\mathbf{U}^j$'s, we can write the portfolio cash scenarios of equation (42) as

$$\mathbf{R}^{\text{port}} = \mathbf{U}^1 \cdot E^1 + \ldots + \mathbf{U}^n \cdot E^n\,. \qquad (65)$$

Since in this equation all the scenario vectors $\mathbf{U}^j$'s are unitary we are inspired to define the *portfolio risk exposure* as

$$E^{\text{port}} = E^1 + \ldots + E^n\,, \qquad (66)$$

and the *portfolio average unitary scenario vector* as

$$\mathbf{U}^{\text{port}} = \frac{\mathbf{U}^1 \cdot E^1 + \ldots + \mathbf{U}^n \cdot E^n}{E^1 + \ldots + E^n}\,, \qquad (67)$$

so that the risk portfolio cash scenarios can all be written to be proportional to the portfolio risk exposure:

$$\mathbf{R}^{\text{port}} = \mathbf{U}^{\text{port}} \cdot E^{\text{port}}\,. \qquad (68)$$

Recall that in computing risk scenarios one should also consider the minimum and maximum viable dates and the appropriate inadmissibility mask.

4.2 Combined server/client risk scenarios

We examine in this subsection the combined effects of the server-side evaluation scenarios, described in section 2, and the client-side computation of cash scenarios defined in section 3. Substituting the risk scenarios computed by the server, given by expression (15), into the portfolio cash scenarios, given by equation (40), and using the results on risk exposure of the previous subsection we obtain, for each index scenario i, for bond-like instruments:

$$\begin{aligned} R_r^{(i)} &= \left\{ \left[1 + b \left(\frac{P^{(i)}}{P_r} - 1 \right) \right] X^{(i)} - 1 \right\} E_r \qquad (69) \\ &= \left(X^{(i)} - 1 \right) E_r + \left(\frac{P^{(i)} - P_r}{P_r} \right) b\, X^{(i)}\, E_r\,; \end{aligned}$$

for futures:

$$R_f^{(i)} = \left(\frac{P^{(i)} - P_r}{P_r} \right) b\, X^{(i)}\, E_f\,; \qquad (70)$$

finally, for swap-like contracts:

$$R_a^{(i)} = \left(X^{(i)} - 1 \right) p_u\, E_a + b\, \frac{P^{(i)} - P_r}{N_r}\, E_a\,. \qquad (71)$$

Comparing the equation for bond-like instruments (69) with that for futures-like contracts (69), we can notice the main difference between the two scenarios types. In the simple case when the asset and the portfolio have the same currency, we have $X^{(i)} = 1$ for all i's, so that bond-like and futures-like scenarios are the same. On the other hand, when the two are in different currencies the two formulas differ by the factor

$$\left(X^{(i)} - 1 \right) E_r\,.$$

Hence, this term represents the contribution to the cash scenarios $R_r^{(i)}$'s of bond-like instruments due to the exchange-rate variation between the single-asset currency and the portfolio currency. More explicitly the difference between bond-like and futures-like scenarios can be written as

$$\begin{aligned} R_r^{(i)} - R_f^{(i)} &= \left(X^{(i)} - 1\right) E_r \\ &= \left(x^{(i)} - x\right) P_r\, q\, f, \end{aligned} \tag{72}$$

where $x^{(i)}$ is the i-th simulated currency-exchange rate (already defined in vector notation in equation (33)). Indeed, for futures-like contracts we consider only how much the exchange-rate scenarios affect the daily marginal account. On the other hand, for bond-like instruments we have invested a lump sum of money in the asset currency so that the exchange-rate scenarios need to be considered.

Notice that, for bond-like instruments and swap-like contracts with $p_u \neq 0$, even in the special case, however rare, in which all simulations $P^{(i)} = P_r$ we still have non-zero risk scenarios for assets with a currency different from the portfolio one.

For stress-test scenarios we can combine equations (20) and (47) to obtain, for bond-like instruments:

$$\begin{aligned} R_r^{(s)} &= \left\{ \left[1 + b \left(\frac{P^{(s)}}{P_r} - 1 \right) \right] X^{(s)} - 1 \right\} E_r \\ &= \left(X^{(s)} - 1\right) E_r + \left(\frac{P^{(s)} - P_r}{P_r} \right) b\, X^{(s)}\, E_r\,; \end{aligned}$$

for futures:

$$R_f^{(s)} = \left(\frac{P^{(s)} - P_r}{P_r} \right) b\, X^{(s)}\, E_f\,; \tag{73}$$

finally, for swap-like contracts:

$$R_a^{(s)} = \left(X^{(s)} - 1\right) p_u\, E_a + b\, \frac{P^{(s)} - P_r}{N_r}\, E_a\,. \tag{74}$$

We can apply the same remarks of the previous paragraphs for risk also to stress-test scenarios derived here.

Finally, for bid/ask-liquidity scenarios, combining formulas (23) and (27) with (49) and (50), for any given market condition, we obtain for *long* cash liquidity scenarios,

$$\begin{aligned} R_{(r,f)}^{-} &= \left[b \cdot \frac{P_r - P^{-}}{P_r} \right]_0^1 \cdot E_{(r,f)} && \text{bond-like and futures-like,} \\ R_a^{-} &= \left[b \cdot \frac{P_r - P^{-}}{N_r} \right]_0 \cdot E_a && \text{swap-like;} \end{aligned} \tag{75}$$

on the other hand, for *short* cash liquidity scenarios we have,

$$\begin{aligned} R_{(r,f)}^{+} &= b \cdot \left[\frac{P^{+} - P_r}{P_r} \right]_0 \cdot E_{(r,f)} && \text{bond-like and futures-like,} \\ R_a^{+} &= b \cdot \left[\frac{P^{+} - P_r}{N_r} \right]_0 \cdot E_a && \text{swap-like.} \end{aligned} \tag{76}$$

Notice how the definitions of risk exposure (58) allowed us to write all these formulas in a simplified form.

4.3 Risk computations for a single asset

Comparing two assets, or when making a decision to add a new asset to an existing portfolio, we may ask ourselves what are the historical risk simulations of an asset when taken by itself. Therefore we can set up a tiny portfolio containing only the asset under scrutiny and compute its risk. We show in this subsection what are sensible values for the portfolio parameters in order to compute meaningful risk figures for the single asset.

A risk measure $\mathcal{R}$ in general depends on the asset evaluation, the simulated scenario variations, the currency-exchange rate, the quantity, and the price factor. Since the computation of $\mathcal{R}$ may vary according to its side in a portfolio, long or short, the quantity parameter q must be set as,

$$\begin{aligned} q &= +1 \quad \text{for the long side}\,, \\ q &= -1 \quad \text{for the short side}\,. \end{aligned} \tag{77}$$

The portfolio currency is always assumed to equal the instrument currency so that the exchange rate is always assumed to be unitary,

$$x = 1\,, \tag{78}$$

and the price factor f is taken, as usual, from the risk server. Finally the client price P_u (or p_u) should be replaced by the reference price P_r (or p_r) computed by the server, i.e. we set

$$P_u = P_r \qquad \text{and} \qquad p_u = p_r\,. \tag{79}$$

Once the portfolio risk measure $\mathcal{R}$ has been evaluated, the asset risk can be computed as,

$$\begin{cases} \dfrac{\mathcal{R}\,(P_r,\ldots)}{P_r} & \text{bond-like and futures-like,} \\ \mathcal{R}\,(p_r,\ldots) & \text{swap-like}\,, \end{cases} \tag{80}$$

where only the evaluation dependence on the risk measure has been shown. Notice that formula (80) holds for a general risk horizon.

5 Simulations for different risk horizons

In sections 2 and 3 we described a framework for the computation of portfolio risk, stress tests, and bid/ask liquidity simulations using a client/server architecture. The main advantage of this set up is that computationally-intensive pricing-function evaluations are worked out on specialized hardware periodically. In this way the client computers do not need huge computing power (for example, portfolio simulations could run on a spreadsheet). In this framework risk simulations can be computed on any single-period length, however the choice of a risk horizon of one business day seems the most sensible. Sometimes, however, professionals need the computation of risk figures on different time horizons such as one week or one month. In this section we describe how to use the one-day risk server simulations to approximate risk scenarios when the required time horizon is longer than one day.

When a time horizon different from one day is needed one could, in theory, compute a new set of simulated scenarios using a different simulation date d' in equation (11). However, this task would be very expensive in terms of both computational time and database storage and greatly increases the overall costs. Hence, most users are happy with cheaper results even thought they provide approximate figures. In table 2 we list the most commonly needed time horizons together with the corresponding number of business days they refer to.

The reader should not confuse the weekly scenarios described in subsection 3.5, where we change the observation frequency from one day to one week, with the scaled-horizon scenarios, described in this section, where we artificially scale the future one-day risk horizon.

Missing market quotes Because of financial market innovations for a growing number of financial instruments, such as exotic options or non-standard swap contracts, it may be hard to find a quote for the market price or an updated estimate for the net present value. In this case the user is usually happy to work with the server-side model evaluations even if they were computed at an earlier date. Therefore for bond-like instruments and futures contracts we use the generic term *market price*, denoted with the symbol P_m, either for the risk-server model price P_r or for the user price P_u. Similarly, for swap-like contracts we use the generic unitary market evaluation p_m either to mean the client-side evaluation p_u or the risk-server model evaluation P_r/N_r.

5.1 Pure scaling of time horizon

To start we consider the simplified case of a single asset with the same currency as the portfolio, for instance a bond-like instrument. Also we assume to have available the set of single-period, i.e. daily, variations $S^{(i)}$'s obtained using equations (15). With the help of expression (59), we can compute the unitary cash daily scenarios as

$$\mathbf{U} = \mathbf{S}_r - 1\,.$$

Given a new time horizon of h business days, we can make the popular assumption that the cash scenarios

Risk horizon	h
one day	1
one week	5
two weeks	10
one month	20
three months	62
six months	125
one year	250

Table 2: List of popular time horizons together with the associated number of business days h.

for a time horizon of h days are scaled as the square root[6] of h, so that the h-day cash scenarios $\widetilde{\mathbf{U}}$'s are given by

$$\widetilde{\mathbf{U}}_r = (\mathbf{S}_r - 1) \cdot \sqrt{h}\,.$$

The same result could be obtained assuming an expression for the scaled unitary scenario vector similar to (59), i.e.

$$\widetilde{\mathbf{U}}_r = \widetilde{\mathbf{S}}_r - 1\,, \tag{81}$$

however with the h-day scaled variations $\widetilde{\mathbf{S}}$ given by

$$\widetilde{\mathbf{S}}_r = 1 + (\mathbf{S}_r - 1) \cdot \sqrt{h}\,. \tag{82}$$

Equation (82) in turn is consistent (see expression (15)) with the scaled simulated prices $\widetilde{P}^{(i)}$'s given by

$$\widetilde{P}^{(i)} = P_m \left[1 + \left(S_r^{(i)} - 1\right) \cdot \frac{\sqrt{h}}{b}\right]\,, \tag{83}$$

for each index scenario i.

Similarly, the scaled variations for futures-like instruments should be computed as

$$\begin{aligned}\widetilde{\mathbf{S}}_f &= \frac{P_m + P_m\,\mathbf{S}_f\,\sqrt{h}}{P_r} - 1\\ &= \mathbf{S}_f\sqrt{h}\end{aligned}$$

and those for swap-like instruments as

$$\begin{aligned}\widetilde{\mathbf{S}}_a &= \left(p_m + \mathbf{S}_a\,\sqrt{h}\right) - p_m\\ &= \mathbf{S}_a\,\sqrt{h}\,.\end{aligned}$$

Note that in all cases the scaled scenarios $\widetilde{\mathbf{S}}$ do not depend on the market price P_m nor on the market unitary net present value p_m.

Given the rescaled scenarios $\widetilde{\mathbf{S}}$'s the rescaled simulated prices can be computed by inverting equations (15). Given a simulation index i, for futures-like instruments we have

$$\widetilde{P}^{(i)} = P_m \left(1 + \frac{\widetilde{S}_f^{(i)}}{b}\right) \tag{84}$$

and for swap-like products

$$\widetilde{p}^{(i)} = p_m + \frac{\widetilde{S}_a^{(i)}}{b} \tag{85}$$

both similar to equation (83) for bond-like instruments.

Rescaled exchange-rate variations In order to compute portfolio risk simulations we also need to scale the currency-exchange rate variations. Inspired by equation (82) we write

$$\widetilde{\mathbf{X}} = 1 + (\mathbf{X} - 1) \cdot \sqrt{h}\,. \tag{86}$$

The portfolio historical simulations on a scaled time horizon can then be computed still using equations (40), however replacing $\mathbf{X}$ with $\widetilde{\mathbf{X}}$ for each variation type.

[6] The $\sqrt{h}$ scaling is consistent with the time-scaling of a Brownian motion.

5.2 Horizon scaling for instruments with bounded prices

The time scaling has at-least an undesired side effect: the resulting scaled scenario may not satisfy some bounding constraints of the original pricing function. In other words, for financial reasons, it is possible that the asset price is constrained between a floor value F and a cap value C (with $F \leq C$); however the scaled price corresponding to a certain variation, see equations (83)–(85), may not always satisfy these bounding conditions. For example, a vanilla put option cannot be worth more than its strike: in this case C is the option strike. Alternatively, a zero-coupon bond cannot be worth more than its redemption, so that C in this case is the redemption value.

Recalling definition (26), we define the *strip* operator for a generic quantity Y, a cap C, and floor F, as

$$[\,Y\,]_F^C = \min\left[C\,;\max\left(Y\,;F\right)\right]\ . \qquad (87)$$

We apply this operator to the price obtained from a rescaled variation so that we can ensure the price bounding between the floor F and the cap C.

Formally, in order to retain the bounding features of the original pricing functions even after the rescaling, we need to apply the strip operator to the simulated scaled price and then compute the corresponding scenarios. Therefore we define the rescaled historical scenarios $\widetilde{\mathbf{S}}^{(i)}$ as,

$$\begin{aligned} \widetilde{S}_r^{(i)} &= 1 + b\cdot\left(\frac{\left[\widetilde{P}^{(i)}\right]_F^C}{\widetilde{P}_m} - 1\right) && \text{bond-like,}\\ \widetilde{S}_f^{(i)} &= b\cdot\left(\frac{\left[\widetilde{P}^{(i)}\right]_F^C}{\widetilde{P}_m} - 1\right) && \text{futures-like,}\\ \widetilde{S}_a^{(i)} &= b\cdot\left(\left[\tilde{p}^{(i)}\right]_F^C - \tilde{p}_m\right) && \text{swap-like.} \end{aligned} \qquad (88)$$

where the market price $\widetilde{P}_m$ and the unitary evaluation $\tilde{p}_m$ are constrained to be always between C and F:

$$\widetilde{P}_m = [\,P_m\,]_F^C \qquad \text{and} \qquad \tilde{p}_m = [\,p_m\,]_F^C\ . \qquad (89)$$

As already noted in these equations P_m (or p_m) can either be the user values P_u (or p_u) or the risk-server model values P_r (or p_r) depending on the following use cases:

1. when computing risk for single assets it should be the model price P_r (or p_r) in the asset currency obtained from the risk server
2. for portfolio risk functions it should be the user price P_u (or p_u), always expressed in the asset currency

Furthermore note that since the cap and the floor values are expressed in the asset currency, also P_m (or p_m) should be expressed in that currency.

Standard values for caps and floors. The values for C and F should be defined in the risk server and such definitions should depend on the asset type. Generally, the floor value of an instrument should be set according to the following rules:

F undefined	swap-like,
$F = 0$	bond-like and futures-like.

On the other hand, the cap value should follow these rules:

C = redemption	for zero-coupon bonds ,
C = strike · parity	for put options ,
$C = F_X$ · strike · parity	for forex put options ,
C is undefined	for all other instruments .

The risk clients should set $F = -\infty$ (or $C = +\infty$) when F (or C) is not. For foreign-exchange options (forex options in the table) F_X is the currency-exchange rate between natural option currency, i.e. the so-called *"to currency"*, and the instrument currency. For example, if we consider a put option on the EUR-USD currency exchange quoted in USD, we have F_X=1. However if the option were quoted in GBP, then F_X would be the GBP-USD exchange rate.

5.3 Scaling portfolio scenarios

We derived in equations (88) the rescaling of historical-simulation variations of a single asset. When considering the asset as a part of a portfolio we also need to rescale the foreign-exchange rate scenarios between the asset currency and the portfolio currency as shown in equation (86). Using for the simulated exchange rates a procedure similar to the one for the assets, we need to ensure that the rescaled exchange rates do not reach negative values. Hence we enforce that the scenario variations have a floor with a zero value, i.e.:

$$\widetilde{\mathbf{X}} = \frac{\left[x + x\,(\mathbf{X} - 1)\,\sqrt{h}\,\right]_0}{x} \tag{90}$$

$$= \left[1 + (\mathbf{X} - 1)\,\sqrt{h}\right]_0 , \tag{91}$$

with the element-by- element definition of the vector strip operator. Also, in order to compute the rescaled historical simulation of portfolio scenarios we need to modify expression (40) for **R** to account for rescaling and bounding. Therefore the rescaled portfolio cash scenarios should be computed using the following cash scenarios

$$\begin{aligned} \widetilde{\mathbf{R}}_r &= \left(\widetilde{\mathbf{S}}_r \cdot \widetilde{\mathbf{X}} - 1\right) \cdot E_r && \text{relative}\,, \\ \widetilde{\mathbf{R}}_f &= \left(\widetilde{\mathbf{S}}_r \cdot \widetilde{\mathbf{X}}\right) \cdot E_f && \text{futures}\,, \\ \widetilde{\mathbf{R}}_a &= \left[\left(p_m + \widetilde{\mathbf{S}}_a\right) \cdot \widetilde{\mathbf{X}} - p_m\right] \cdot E_a && \text{absolute}\,. \end{aligned} \tag{92}$$

for each asset and then composed together as explained in subsection 3.2.

6 Summary and conclusions

In this paper we described the client/server approach to analyze portfolio risk scenarios, stress-test simulations, and bid/ask liquidity spreads. We focused on the risk-client tasks, specifically on how the risk clients process the simulated price variations computed by the risk server to obtain portfolio cash simulations.

We have also shown how the dual approach is very efficient and lowers the computational costs, since the numerically intensive calculations of each single asset are performed for all clients and all portfolios at once. The algorithms used are very similar for risk scenarios, stress-test simulations, and bid/ask liquidity spreads. For risk scenarios, however, we need to also keep track of the possible missing scenario dates. In order to perform portfolio cash simulations, the risk clients handle bond-like, swap-like, and futures-like instruments in different ways. The risk clients then combine the generated scenarios to compute portfolio cash scenarios. Moreover we have shown that the risk, stress, and liquidity cash simulations provide a natural definition of risk exposure. In a separate paper, see reference [2], we show how to generalize the risk-scenario computations using unified formulas for the three cases of bond-like, futures-like, and swap like asset types.

Finally, we described how the risk clients, starting from daily risk simulations, can approximately compute portfolio risk simulations over a wider risk horizon preserving the bounding properties of each instrument.

One of the great features of the approach described in this paper is that it scales very well with the number of portfolios handled. Indeed it is possible to compute the risk analysis for several thousands of portfolio such as those handled by portfolio-analytics web sites such as that reported in reference [5].

References

[1] Dario Cintioli and Marco Marchioro. Liquidity risk simulations. In *The Simulation Model*, Quantitative Research Series. StatPro website, June 2009. Previews available at `http://marchioro.org`. 6

[2] Marco Marchioro and Davide Borrello. A unified approach to the computation of risk scenarios. In *Internal Papers*, Quantitative Research Series. StatPro website, August 2013. Previews are not available. 21

[3] Marco Marchioro and Dario Cintioli. The foundations of the statpro simulation model. In *The Sim-*

ulation Model, Quantitative Research Series. StatPro website, October 2007. Previews available at `http://marchioro.org`. 2, 3, 10

[4] Attilio Meucci. *Risk and Asset Allocation*. Springer, 2005. 4

[5] StatPro Revolution. A cloud-based portfolio analytics service. `http://statpro.com/revolution`. 21